LECTURES ON ELECTRO-DYNAMICS

J. R. Oppenheimer

documents on modern physics

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Lectures on Electrodynamics

Documents on Modern Physics

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Preface

"His lectures were a great experience, for experimental as well as theoretical physicists. In addition to a superb literary style, he brought to them a degree of sophistication in physics previously unknown in the United States." So wrote Hans Bethe of Robert Oppenheimer's qualities as a teacher during the decade prior to World War II. It was an exciting decade for physics, and Oppenheimer, holding a joint appointment at the University of California and the California Institute of Technology, was attracting a group of students destined to contribute to the great transformation which would soon propel American physics into the front rank. One of these students, Robert Serber, who followed Oppenheimer on his annual trek between Berkeley and Pasadena, has said of one of his courses: "It was an inspirational as well as an educational achievement. He transmitted to his students a feeling of the beauty of the logical structure of physics and an excitement about the development of physics. Almost everyone listened to the course more than once."

Perhaps classical electrodynamics has by now lost some of the excitement it had in the thirties, when it was still viewed as the primary foundation and model for quantum electrodynamics. We have become so sophisticated nowadays that the theorists among us, when we think of electrodynamics, think of threshold theorems, off-shell amplitudes and dispersion relations—almost never of little radiating balls of mass and charge. And yet classical electrodynamics has a standard place in our curriculum. The lecture notes which comprise this book have been reproduced in ditto-copy form three times since they were first written down: in Berkeley in 1939, then at the University of Chicago in 1947 and again at the University of Colorado in 1949. I am grateful to Dr. Richard Akerib for providing a personal copy of the latter set from which the present, and first published edition, has been prepared. I wish also to thank Mrs. Oppenheimer for granting the publisher permission to bring these lectures to a wider audience.

Aside from some updatings of notation and corrections of obvious slips of the pen, very little has been changed from the original notes. I have improved the English in a number of places and rewritten occasional paragraphs for clarity. In chapter 1, sections 9 and 10, I have simplified some derivations. There are also a few editorial remarks scattered through the book. Only rarely does the text itself sound dated, such as at the end of the paragraph preceding Eq. (11.8) and the brief excursion into angle-and-action-variable perturbation theory in chapter 1, section 17. I have let such vignettes stand unimpaired.

The material is as relevant now as when the lectures were first given. Problems are approached in a direct manner and the transition from theory to application occurs quickly and repeatedly. Basic difficulties are not avoided but are presented in a manner which is both rational and uncomplicated. The viewpoint is by no means confined to the purely classical theory. Quantum ideas are introduced at every relevant occasion, and relativity theory is developed in a delightfully revealing way. The prerequisites for reading these notes are modest. The student should have had a course in electromagnetic theory and some acquaintance with elementary quantum theory. This will suffice to enable him to share some of the same joys as an earlier generation of students.

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CHAPTER I

Maxwell's Theory

1 INTRODUCTION: DEFINITION OF THE FIELDS

Electrodynamics is a field theory. It deals with the electric and magnetic fields, supposed measurable at any point $P(x, y, z)$ in space and at any time, t. We shall be concerned with fields measured in the absence of dielectrics and diamagnetics: the electric field $\mathbf{E}(x, y, z, t)$, the magnetic field $\mathbf{H}(x, y, z, t)$: When we have to discuss dielectric effects we shall try to understand them atomically; that is, in terms of the fields due to the charges in the medium. Then we shall have to distinguish, by the type of measurement involved, $\mathbf{D}$ from $\mathbf{E}$, $\mathbf{B}$ from $\mathbf{H}$. These are all vectors and functions of x, y, z, and t.

Definition of the fields: We take a test body, of volume V, uniform charge density ϱ, and mass M. Let it be very nearly at rest, and let it occupy the volume V about P for a time τ about t. The electric field, averaged over V and τ is then defined by

$$\mathbf{p}(t + \tau) - \mathbf{p}(t) = V\varrho\tau\mathbf{E}$$

where $\mathbf{p} = M\mathbf{v}$ is the momentum, $\mathbf{v}$ the velocity of the test body. If $\mathbf{p}$ is to be measurable and $\mathbf{v}$ so small that magnetic forces can be neglected, M must be large. The precautions to make the measurement valid are:

1) $\mathbf{v} \to 0$ no magnetic forces.

$$\left.\begin{array}{ll} 2) & \tau \to 0 \\ 3) & V \to 0 \end{array}\right\}\ \text{to the time and point at which } \mathbf{E} \text{ was measured.}$$

4) $\varrho V \to 0$ to eliminate electromagnetic self-action effects of the test body.

In electron theory, ϱV cannot be less than the electronic charge e. In quantum theory $\mathbf{p}$ cannot be measured without sacrificing the knowledge of the test body's position. When atomic and quantum effects are important, difficulties in definition of the field arise. The quantum limitations are well understood and form the physical basis of the breakdown of classical electromagnetic theory.

The analogous definition of **H** would use a distribution μ of magnetic poles:

$$\mathbf{p}(t + \tau) - \mathbf{p}(t) = V\mu\tau\mathbf{H}$$

and the same precautions as for **E**. We have no single poles, but we use the ends of long needles. The **H** so measured gives a force on a moving charge e, the Lorentz law of force,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = e\left\{\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{H})\right\}$$

We can use this to extend the definition of **H**.

2 MAXWELL'S EQUATIONS

In the Gaussian system of units, Maxwell's equations are

$$\nabla \cdot \mathbf{E} = 4\pi\varrho \tag{2.1}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \tag{2.3}$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + 4\pi\mathbf{j} \tag{2.4}$$

where ϱ and **j** are the charge and current densities. The first three equations are the expressions of experimental facts in differential form. (2.1) is Coulomb's law for electric charges; (2.2) is Coulomb's law for magnetic poles, supplemented by the fact that no free magnetic poles exist in nature; (2.3) is Faraday's law of induction. (2.4) is a generalization of Ampere's law

$$\nabla \times \mathbf{H} = 4\pi\mathbf{j}$$

by the addition of the term $\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$, which is called Maxwell's displacement current. Maxwell added this term since otherwise the equation leads to

$$\nabla \cdot \mathbf{j} = 0$$

and this is not true for non-steady flow of current. When the term is included, we get

$$\nabla \cdot \mathbf{j} = -\frac{1}{c}\frac{\partial \varrho}{\partial t} \tag{2.5}$$

and if we write

$$\mathbf{j} = \frac{1}{c}\mathbf{v}\varrho$$

we have the usual equation of continuity

$$\nabla \cdot (\varrho \mathbf{v}) + \frac{\partial \varrho}{\partial t} = 0$$

Maxwell's additional term is also necessary to have **E** and **H** satisfy the wave equation in free space. If the term is not included, we get

$$\begin{aligned} \varDelta \mathbf{E} &= 0 \\ \varDelta \mathbf{H} &= 0 \end{aligned} \qquad (\varDelta \equiv \nabla \cdot \nabla)$$

which do not admit plane wave solutions and hence contradict experience.

EXERCISE 1 Describe an experiment whereby the existence of Maxwell's displacement current is verified directly.

EXERCISE 2 Show that the equation

$$\nabla \times \mathbf{H} = 4\pi \mathbf{j}$$

follows directly from the equivalence between a current circuit and a suitable magnetic shell.

3 SOLUTION OF THE EQUATIONS IN FREE SPACE

At first sight it may be thought that from Maxwell's equations given above and from the Lorentz force

$$\mathbf{f} = \varrho \mathbf{E} + \mathbf{j} \times \mathbf{H} \tag{3.1}$$

($\mathbf{f}$ = force density) we could compute the charge and current distribution ϱ, $\mathbf{j}$ and the fields **E** and **H** from given initial conditions. But besides the mathematical difficulty, there is the yet unsolved problem of how the self-field of a charge affects its motion. Hence, we have to limit ourselves to the less ambitious problem of calculating the field produced by a given charge and current distribution, and conversely the charge and current from a given field. In this way we may approach the general solution by successive approximation.

First we shall consider the solution of Maxwell's equations in free space. Putting $\varrho = 0$, and $\mathbf{j} = 0$, we have

$$\nabla \cdot \mathbf{E} = 0 \tag{3.2}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{3.3}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \tag{3.4}$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{3.5}$$

Taking the curl of the last two equations and noting that

$$\nabla \times \nabla \times = \nabla \nabla \cdot - \Delta \tag{3.6}$$

we get

$$\Box \mathbf{E} = 0 \tag{3.7}$$

$$\Box \mathbf{H} = 0 \tag{3.8}$$

by using the first two equations. Here

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta \tag{3.9}$$

and is known as the d'Alembertian operator.

Suppose each component of **E**, **H** is a function of z and t only. Then

$$\frac{1}{c^2}\frac{\partial^2 E_i}{\partial t^2} - \frac{\partial^2 E_i}{\partial z^2} = 0; \qquad (i = x, y, z) \tag{3.10}$$

$$\frac{1}{c^2}\frac{\partial^2 H_i}{\partial t^2} - \frac{\partial^2 H_i}{\partial z^2} = 0 \tag{3.11}$$

The solutions of these equations are

$$E_i = f_i(z + ct) + F_i(z - ct) \tag{3.12}$$

$$H_i = g_i(z + ct) + G_i(z - ct) \tag{3.13}$$

In order that they satisfy Maxwell's equation, we have from (3.2) and (3.3)

$$\frac{\partial E_z}{\partial z} = 0; \qquad \frac{\partial H_z}{\partial z} = 0$$

and hence from (3.12) and (3.13), $E_z = 0$, $H_z = 0$, apart from a constant field which is of no interest here, which shows that electromagnetic waves are transverse waves. In free space the direction of the electric and magnetic vectors is always perpendicular to the direction of propagation of the wave. Let **E** be along the x-axis; then $E_z = E_y = 0$. From (3.4) and (3.5) we get

$$\frac{\partial E_x}{\partial z} = -\frac{1}{c}\frac{\partial H_y}{\partial t}; \qquad -\frac{\partial H_y}{\partial z} = \frac{1}{c}\frac{\partial E_x}{\partial t}$$

and $H_x = H_z = 0$. Thus **E**, **H**, **z**, where **z** is the direction of propagation, form a right-handed orthogonal system of vectors, and we have

$$E_x = f(z + ct) + F(z - ct)$$

$$H_y = -f(z + ct) + F(z - ct)$$

as solutions of Maxwell's equations.

Let us consider the special case of monochromatic waves. Then

$$f(z + ct) = R\{Ae^{ik(z+ct)}\}$$

where $R\{\ \}$ denotes the real part. If **n** is a unit vector in the direction of propagation, then

$$\mathbf{k} = k\mathbf{n} = \frac{2\pi\nu}{c}\mathbf{n} = \frac{2\pi}{\lambda}\mathbf{n}$$

is called the propagation vector. Let us introduce unit vectors $\boldsymbol{\varepsilon}^1$, $\boldsymbol{\varepsilon}^2$ so that $\boldsymbol{\varepsilon}^1$, $\boldsymbol{\varepsilon}^2$, **n** form a right-handed system of orthogonal unit vectors. Then for a monochromatic wave propagating in the direction **n**,

$$\mathbf{E} = \boldsymbol{\varepsilon}^1\{a^1 e^{i(\mathbf{k}\cdot\mathbf{n} - \omega t)} + \bar{a}^1 e^{-i(\mathbf{k}\cdot\mathbf{n} - \omega t)}\}$$

$$\mathbf{H} = \boldsymbol{\varepsilon}^2\{b^2 e^{i(\mathbf{k}\cdot\mathbf{n} - \omega t)} + \bar{b}^2 e^{-i(\mathbf{k}\cdot\mathbf{n} - \omega t)}\}$$

Here we have written $\omega = kc$. We can obtain the most general expression for **E** and **H** in free space by superposition of plane waves, thus:

$$\mathbf{E} = \sum_{\lambda}\int d\mathbf{k}\, \boldsymbol{\varepsilon}_{\mathbf{k}}^{\lambda}\{a_{\mathbf{k}}^{\lambda} e^{i(\mathbf{k}\cdot\mathbf{n} - \omega t)} + \bar{a}_{\mathbf{k}}^{\lambda} e^{-i(\mathbf{k}\cdot\mathbf{n} - \omega t)}\} \tag{3.14}$$

$$\mathbf{H} = \sum_{\lambda}\int d\mathbf{k}\, \boldsymbol{\varepsilon}_{\mathbf{k}}^{\lambda}\{b_{\mathbf{k}}^{\lambda} e^{i(\mathbf{k}\cdot\mathbf{n} - \omega t)} + \mathbf{b}_{\mathbf{k}}^{\lambda} e^{-i(\mathbf{k}\cdot\mathbf{n} - \omega t)}\} \tag{3.15}$$

with

$$d\mathbf{k} = dk_x\, dk_y\, dk_z$$

The constants $b_{\mathbf{k}}^{\lambda}$ are simply related to $a_{\mathbf{k}}^{\lambda}$ since we have the relation (3.4)

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$

Now

$$\nabla \times (\boldsymbol{\varepsilon}_{\mathbf{k}}^{\lambda} e^{i\mathbf{k}\cdot\mathbf{n}}) = ie^{i\mathbf{k}\cdot\mathbf{n}}[\mathbf{k} \times \boldsymbol{\varepsilon}_{\mathbf{k}}^{\lambda}]$$

Hence

$$\nabla \times \mathbf{E} = i\sum_{\lambda}\int d\mathbf{k}[\mathbf{k} \times \boldsymbol{\varepsilon}_{\mathbf{k}}^{\lambda}]\,\{a_{\mathbf{k}}^{\lambda} e^{i(\mathbf{k}\cdot\mathbf{n} - \omega t)} - \bar{a}_{\mathbf{k}}^{\lambda} e^{-i(\mathbf{k}\cdot\mathbf{n} - \omega t)}\}$$

$$-\frac{1}{c}\frac{\partial H}{\partial t} = \frac{i}{c}\sum_{\lambda}\int d\mathbf{k}\,\omega\boldsymbol{\varepsilon}_{\mathbf{k}}^{\lambda}\{b_{\mathbf{k}}^{\lambda} e^{i(\mathbf{k}\cdot\mathbf{n} - \omega t)} - \mathbf{b}_{\mathbf{k}}^{\lambda} e^{-i(\mathbf{k}\cdot\mathbf{n} - \omega t)}\}$$

Remembering that

$$\mathbf{k} \times \boldsymbol{\varepsilon}_{\mathbf{k}}' = k\boldsymbol{\varepsilon}_{\mathbf{k}}^{2}$$
$$\mathbf{k} \times \boldsymbol{\varepsilon}_{\mathbf{k}}^{2} = -k\boldsymbol{\varepsilon}_{\mathbf{k}}'$$

we get

$$a_{\mathbf{k}}^{1} = b_{\mathbf{k}}^{2}$$
$$a_{\mathbf{k}}^{2} = -b_{\mathbf{k}}^{1} \tag{3.16}$$

and the complex conjugates of these equations.

The fields can be expanded in terms of any other complete set of orthogonal functions, but the above Fourier expansion is convenient as it is simple analytically and we shall see later that the energy and momentum of radiation can be expressed simply in terms of the coefficients $a_{\mathbf{k}}^{\lambda}$. Also, though the field near a radiating charge is very complicated, at large distances away it can be expressed as a sum of plane waves and a coulomb field.

4 APPLICATIONS TO THE SKIN EFFECT AND METALLIC REFLECTION

Consider a semi-infinite conductor. As long as the dimensions of the skin are small compared to the radius of curvature of the wire, we can use the result derived above to study the skin effect in wires. Inside the conductor

$$\mathbf{j} = \frac{1}{c}\sigma\mathbf{E} \tag{4.1}$$

where σ is the conductivity. First let us consider copper since for this metal, both the dielectric constant $\varkappa$ and the permeability μ can be taken as unity.

Maxwell's equations inside the metal are then

$$\nabla \cdot \mathbf{E} = 4\pi\varrho \tag{4.2}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{4.3}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \tag{4.4}$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi\sigma}{c}\mathbf{E} \tag{4.5}$$

Let us choose our axes so that the xy plane is parallel to the boundary surface of the metal, and let radiation of frequency ω and plane polarized in the x direction be incident normally on the boundary. Then

$$E_x(z, t) = f(z)\, e^{-i\omega t}$$

$$H_y(z, t) = g(z)\, e^{-i\omega t}$$

and we shall use the convention of taking the real part of **E** and **H** as the actual field. Assume

$$\varrho = F(z)\, e^{-i\omega t}$$

Then the conservation equation

$$\frac{1}{c}\frac{\partial \varrho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

and (4.1) give

$$-\frac{1}{c} i\omega F(z)\, e^{-i\omega t} + \frac{\sigma}{c} \nabla \cdot \mathbf{E} = 0$$

and using (4.2) we have

$$-\frac{i\omega}{c} F(z)\, e^{-i\omega t} + \frac{4\pi\sigma}{c} F(z)\, e^{-i\omega t} = 0$$

and hence

$$F(z) = 0$$

$$\varrho = 0$$

and

$$\nabla \cdot \mathbf{E} = 0 \tag{4.6}$$

Taking the curl of (4.4) and using (4.5) we have

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{4\pi\sigma}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

and using (3.6) and (4.6) we get

$$\Delta \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{4\pi\sigma}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

$$\frac{d^2 f}{dz^2} = \left(-\frac{\omega^2}{c^2} - \frac{4\pi i\sigma\omega}{c^2}\right) f \tag{4.7}$$

Taking

$$f = Ae^{(i\varkappa - \eta)z}$$

we have

$$(i\varkappa - \eta)^2 = -\frac{\omega^2}{c^2}\left(1 + \frac{4\pi i\sigma}{\omega}\right)$$

whence

$$\varkappa = \frac{\omega}{c}\left[\frac{1}{2}\left\{1 + \left(\frac{4\pi\sigma}{\omega}\right)^2\right\}^{1/2} + \frac{1}{2}\right]^{1/2}$$

$$\eta = \frac{\omega}{c}\left[\frac{1}{2}\left\{1 + \left(\frac{4\pi\sigma}{\omega}\right)^2\right\}^{1/2} - \frac{1}{2}\right]^{1/2}$$

which for

$$\frac{4\pi\sigma}{\omega} \gg 1$$

yields

$$\left.\begin{array}{l}\varkappa \approx \\ \eta \approx\end{array}\right\} \frac{\sqrt{2\pi\sigma\omega}}{c}$$

$1/\eta$ is the distance from the boundary to the point in the metal where the strength of the field drops to $1/e$ of the value at the boundary, and hence it is a measure of the thickness of the skin when an alternating current of frequency $\omega/2\pi$ is flowing in the metal.

Exercise 3 Find the value of ω for which $1/\eta = 10^{-3}$ cm in copper.

For ferromagnetics where μ varies with ω and is not unity, it is found that

$$\left.\begin{array}{l}\varkappa \approx \\ \eta \approx\end{array}\right\} \sqrt{\frac{2\pi\sigma\omega}{c^2}\,\mu(\omega)}$$

Exercise 4 Describe an experiment whereby μ can be measured as a function of ω.

The reflecting power of metals can be found by solving a simple boundary value problem. The procedure is to adjust the amplitude and phases of the reflected and transmitted waves so that **E** and **H** are continuous at the boundary. From (4.4) and (4.5) we see that H also satisfies Eq. (4.7) so that

$$H_y = Be^{(i\varkappa - \eta)z - i\omega t}$$

and then (4.4) gives (with c replaced by c/μ to take permeability into account)

$$\frac{H_y}{E_x} = +\frac{(i\varkappa - \eta)\,c}{i\omega\mu}$$

$$\frac{|H_y|}{|E_x|} = \sqrt{\frac{4\pi\sigma}{\omega\mu}}$$

Let the incident beam be given by

$$E_x = Ae^{-i\omega(t - z/c)}$$

$$H_y = Ae^{-i\omega(t - z/c)}$$

Then the reflected beam will be given by

$$E_x' = A'e^{-i\omega(t - z/c)}$$

$$H_y' = -A'e^{-i\omega(t - z/c)}$$

and the transmitted beam by

$$E_x'' = A''e^{(i\varkappa - \eta)z - i\omega t}$$

$$H_y'' = \frac{(i\varkappa - \eta)\,c}{i\omega\mu} A''e^{(i\varkappa - \eta)z - i\omega t}$$

The condition of continuity of **E** and **H** at the boundary ($z = 0$) gives

$$A + A' = A''$$

$$A - A' = \frac{(i\varkappa - \eta)\,c}{i\omega\mu} A''$$

and from these equations we find the reflecting power r as

$$r = \frac{|A'|^2}{|A|^2} = \left|\frac{1 - \dfrac{c}{\omega\mu}(\varkappa + i\eta)}{1 + \dfrac{c}{\omega\mu}(\varkappa + i\eta)}\right|^2$$

EXERCISE 5 Calculate r for copper ($\mu = 1$) for the value of ω such that $1/\eta = 10^{-3}$ cm.

5 ENERGY AND MOMENTUM OF AN ELECTROMAGNETIC FIELD

The energy and momentum of a test body are well defined. We consider the changes in them caused by the interaction with an electromagnetic field and assign such values of the energy and momentum to the electromagnetic field that the laws of conservation of energy and momentum will be satisfied. From this we see that since only changes in the energy and momentum of an electromagnetic field are defined, they are undetermined by an additive constant. We shall see later that the theory of relativity determines this constant.

First we shall consider the energy. If E and $\mathbf{p}$ are the energy and momentum of a test body, we have

$$\begin{aligned}\Delta E &= \mathbf{F}\cdot\Delta\mathbf{r}\\ &= (\mathbf{F}\cdot\mathbf{v})\,\Delta t\end{aligned}$$

where

$$\mathbf{F} = \varrho V\{\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\}$$

Therefore

$$\begin{aligned}\frac{1}{V}\frac{\Delta E}{\Delta t} &= \varrho\mathbf{E}\cdot\mathbf{v}\\ &= c\mathbf{E}\cdot\mathbf{j}\end{aligned} \tag{5.1}$$

Similarly from

$$\Delta\mathbf{p} = \mathbf{F}\,\Delta t$$

we get

$$\frac{1}{V}\cdot\frac{\Delta\mathbf{p}}{\Delta t} = \varrho\{\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\} \tag{5.2}$$

We shall require that

$$c(\mathbf{E}\cdot\mathbf{j}) + \frac{\partial W}{\partial t} + \nabla\cdot\mathbf{S} = 0 \tag{5.3}$$

where

W = the energy density of the electromagnetic field

$\mathbf{S}$ = the flux of electromagnetic energy

and obtain an expression for W and $\mathbf{S}$ in terms of $\mathbf{E}$ and $\mathbf{H}$. Equation (5.3) is a statement of the law of conservation of energy.

Now from vector analysis we have

$$\mathbf{E}\cdot\nabla\times\mathbf{H} - \mathbf{H}\cdot\nabla\times\mathbf{E} = -\nabla\cdot(\mathbf{E}\times\mathbf{H})$$

Therefore using Maxwell's Eqs. (2.3) and (2.4) we get

$$\mathbf{E}\cdot\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + 4\pi(\mathbf{E}\cdot\mathbf{j}) + \mathbf{H}\cdot\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} + \nabla\cdot[\mathbf{E}\times\mathbf{H}] = 0$$

$$\frac{c}{4\pi}\nabla\cdot[\mathbf{E}\times\mathbf{H}] + c(\mathbf{E}\cdot\mathbf{j}) + \frac{1}{8\pi}\frac{\partial}{\partial t}(E^2 + H^2) = 0$$

Hence

$$W = \frac{1}{8\pi}(E^2 + H^2) \tag{5.4}$$

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H} \tag{5.5}$$

We see that the formula for the energy density W is the same as that in electro- and magnetostatics. On the other hand if we apply the formula for energy flux to the case of uniform static fields, with $\mathbf{E}$ perpendicular to $\mathbf{H}$, we obtain a nonvanishing result. This seems strange, but as $\nabla \cdot \mathbf{S} = 0$, there is no difficulty.

Exercise 6 Consider the case where $\mathbf{E}$ is still perpendicular to $\mathbf{H}$ but they are not uniform. It seems that $\partial W/\partial t$ does not vanish. Is $\nabla \cdot \mathbf{S}$ equal to zero?

For the momentum, we shall require an equation of the form

$$\frac{\partial \mathbf{G}}{\partial t} + \{\varrho\mathbf{E} + [\mathbf{j} \times \mathbf{H}]\} + \nabla \cdot \mathbf{T} = 0 \tag{5.6}$$

which expresses the law of conservation of momentum. We shall obtain the formula

$$\mathbf{G} = \frac{1}{c^2}\mathbf{S} \tag{5.7}$$

for the momentum of the radiation, and shall obtain an expression for Maxwell's "stress tensor" $\mathbf{T}$. Using (2.3) and (2.4) we get

$$\frac{\partial}{\partial t}\left[\frac{1}{4\pi c}\mathbf{E} \times \mathbf{H}\right] = \frac{1}{4\pi c}\left[\frac{\partial \mathbf{E}}{\partial t} \times H\right] + \frac{1}{4\pi c}\left[\mathbf{E} \times \frac{\partial \mathbf{H}}{\partial t}\right]$$

$$= \frac{1}{4\pi}(\nabla \times \mathbf{H}) \times \mathbf{H} - \mathbf{j} \times \mathbf{H} - \frac{1}{4\pi}\mathbf{E} \times (\nabla \times \mathbf{E})$$

$$= \frac{1}{4\pi}\left\{(\mathbf{H} \cdot \nabla)\,\mathbf{H} + (\mathbf{E} \cdot \nabla)\,\mathbf{E} - \frac{1}{2}\nabla(E^2 + H^2)\right\} - \mathbf{j} \times \mathbf{H}$$

From (2.1) and (2.2) we also get

$$\frac{1}{4\pi}\{\mathbf{H}(\nabla \cdot \mathbf{H}) + \mathbf{E}(\nabla \cdot \mathbf{E})\} - \mathbf{E}\varrho = 0$$

Adding this to the above equation, we obtain

$$\frac{\partial}{\partial t}\left\{\frac{1}{4\pi c}\mathbf{E}\times\mathbf{H}\right\}+\varrho\mathbf{E}+\mathbf{j}\times\mathbf{H}$$

$$=\frac{1}{4\pi}\left\{\mathbf{E}(\nabla\cdot\mathbf{E})+(\mathbf{E}\cdot\nabla)\,\mathbf{E}+\mathbf{H}(\nabla\cdot\mathbf{H})+(\mathbf{H}\cdot\nabla)\,\mathbf{H}-\frac{1}{2}\nabla(E^2+H^2)\right\}$$

Writing this in component form, we have

$$\frac{\partial}{\partial t}\left\{\frac{1}{4\pi c}[\mathbf{E}\times\mathbf{H}]_i\right\}+\varrho\mathbf{E}_i+[\mathbf{j}\times\mathbf{H}]_i$$

$$=\frac{1}{4\pi}\sum_j E_i\,\nabla_j E_j+E_j\,\nabla_j E_i+H_i\,\nabla_j H_i+H_j\,\nabla_j H_i-\frac{1}{2}\nabla_i(E^2+H^2)\}$$

$$=\frac{1}{4\pi}\sum_j \nabla_j\left\{E_iE_j+H_iH_j-\frac{1}{2}\delta_{ij}(E^2+H^2)\right\}\quad\text{with}\quad \nabla_j=\frac{\partial}{\partial x^j}$$

Thus we have (5.6), where

$$\mathbf{G}=\frac{1}{c^2}\mathbf{S}=\frac{1}{4\pi c}\mathbf{E}\times\mathbf{H}\tag{5.8}$$

$$T_{ij}=\frac{1}{4\pi}\left\{-E_iE_j-H_iH_j+\frac{1}{2}\delta_{ij}(E^2+H^2)\right\}\tag{5.9}$$

We note that for a plane electromagnetic wave we can write

$$|\mathbf{G}|=\mu c$$

where

$$\mu=\frac{W}{c^2}$$

which is the first indication of the equivalence between mass and energy. Further, in free space $\mathbf{E}$ is perpendicular to $\mathbf{H}$ and $|\mathbf{E}|=|\mathbf{H}|$ so that

$$\mathbf{S}=c\mathbf{n}W$$

$$\mathbf{G}=\frac{\mathbf{n}W}{c}$$

EXERCISE 7 Show that

$$\sum T_{ii}-W=0$$

EXERCISE 8 1) Calculate the force on an antenna radiating 30 kW in a narrow beam. 2) Calculate the velocity of recoil of a Li^8 nucleus when it emits an 18 MV γ-ray. Also calculate the velocity of recoil of an electron if it emitted such a γ-ray.

The diagonal components T_{ii} of the stress tensor $\mathbf{T}$ represent the pressure exerted by the radiation. For radiation in a cavity

$$\langle E_x^2\rangle = \langle E_y^2\rangle = \langle E_z^2\rangle = \tfrac{1}{3}\langle E^2\rangle = \langle H_x^2\rangle = \langle H_y^2\rangle = \langle H_z^2\rangle = \tfrac{1}{3}\langle H^2\rangle$$

$$\langle E_i E_j\rangle = 0, \quad \langle H_i H_j\rangle = 0, \quad \text{for} \quad i \neq j$$

$$\langle T_{ij}\rangle = 0 \qquad i \neq j$$

$$\langle T_{ii}\rangle = \frac{1}{12\pi}\langle E^2\rangle$$

$$\langle \mathbf{S}\rangle = \langle \mathbf{G}\rangle = 0$$

$$W = \frac{1}{4\pi}\langle E^2\rangle$$

$$\langle T_{ii}\rangle = \text{pressure} = \tfrac{1}{3}W$$

So far we have talked only about energy and momentum densities. The total energy and momentum of a field is obtained by integration

$$E = \int d\mathbf{r}\,\frac{(E^2 + H^2)}{8\pi} \qquad (d\mathbf{r} \equiv dx\,dy\,dz) \tag{5.10}$$

$$\mathbf{p} = \int d\mathbf{r}\,\frac{\mathbf{E}\times\mathbf{H}}{4\pi c} \tag{5.11}$$

If we write $\mathbf{E}$ and $\mathbf{H}$ as sums of terms, then E and $\mathbf{p}$ will be double sums, and in general there will be interference, and the expressions will be complicated. However, if Fourier expansion is used, E and $\mathbf{p}$ can be written simply as single sums. To show this, we have (3.14)

$$\mathbf{E} = \sum_\lambda \int d\mathbf{k}\boldsymbol{\varepsilon}_\mathbf{k}^\lambda\,\{a_\mathbf{k}^\lambda e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \bar{a}_\mathbf{k}^\lambda e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\}$$

Let

$$\alpha_\mathbf{k}^\lambda = a_\mathbf{k}^\lambda e^{-i\omega t}$$

Then

$$\mathbf{E} = \sum_\lambda \int d\mathbf{k}\;\boldsymbol{\varepsilon}_\mathbf{k}^\lambda\{\alpha_\mathbf{k}^\lambda e^{i\mathbf{k}\cdot\mathbf{r}} + \bar{\alpha}_\mathbf{k}^\lambda e^{-i\mathbf{k}\cdot\mathbf{r}}\}$$

whence

$$E^2 = \sum_\lambda \sum_{\lambda'} \int d\mathbf{k} \int d\mathbf{k}'\, \boldsymbol{\varepsilon}_\mathbf{k}^\lambda \cdot \boldsymbol{\varepsilon}_{\mathbf{k}'}^{\lambda'} \{\alpha_\mathbf{k}^\lambda \alpha_{\mathbf{k}'}^{\lambda'} e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} + \bar{\alpha}_\mathbf{k}^\lambda \alpha_{\mathbf{k}'}^{\lambda'} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}$$

$$+ \alpha_\mathbf{k}^\lambda \bar{\alpha}_{\mathbf{k}'}^{\lambda'} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} + \bar{\alpha}_\mathbf{k}^\lambda \bar{\alpha}_{\mathbf{k}'}^{\lambda'} e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\}$$

$$\int E^2\, d\mathbf{r} = 8\pi^3 \sum_\lambda \sum_{\lambda'} \int d\mathbf{k} \int d\mathbf{k}'\, \boldsymbol{\varepsilon}_\mathbf{k}^\lambda \cdot \boldsymbol{\varepsilon}_{\mathbf{k}'}^{\lambda'}$$

$$\times \{(\bar{\alpha}_\mathbf{k}^\lambda \alpha_{\mathbf{k}'}^{\lambda'} + \alpha_\mathbf{k}^\lambda \bar{\alpha}_{\mathbf{k}'}^{\lambda'})\, \delta(\mathbf{k}-\mathbf{k}') + (\alpha_\mathbf{k}^\lambda \alpha_{\mathbf{k}'}^{\lambda'} + \bar{\alpha}_\mathbf{k}^\lambda \bar{\alpha}_{\mathbf{k}'}^{\lambda'})\, \delta(\mathbf{k}+\mathbf{k}')\}$$

$$= 8\pi^3 \sum_\lambda \sum_{\lambda'} \int d\mathbf{k}\{\boldsymbol{\varepsilon}_\mathbf{k}^\lambda \cdot \boldsymbol{\varepsilon}_\mathbf{k}^{\lambda'}(\bar{\alpha}_\mathbf{k}^\lambda \alpha_\mathbf{k}^{\lambda'} + \alpha_\mathbf{k}^\lambda \bar{\alpha}_\mathbf{k}^{\lambda'}) + \boldsymbol{\varepsilon}_\mathbf{k}^\lambda \cdot \boldsymbol{\varepsilon}_{-\mathbf{k}}^{\lambda'}(\alpha_\mathbf{k}^\lambda \alpha_{-\mathbf{k}}^{\lambda'} + \bar{\alpha}_\mathbf{k}^\lambda \bar{\alpha}_{-\mathbf{k}}^{\lambda'})\}$$

Now

$$\boldsymbol{\varepsilon}_\mathbf{k}^\lambda \cdot \boldsymbol{\varepsilon}_\mathbf{k}^{\lambda'} = \delta_{\lambda\lambda'}$$

and we can choose $\boldsymbol{\varepsilon}_{-\mathbf{k}}^1$ so that

$$\boldsymbol{\varepsilon}_{-\mathbf{k}}^1 = \boldsymbol{\varepsilon}_\mathbf{k}^1$$

Then

$$\boldsymbol{\varepsilon}_{-\mathbf{k}}^2 = -\boldsymbol{\varepsilon}_\mathbf{k}^2$$

Using these facts and replacing the $\alpha_\mathbf{k}^\lambda$ by the $a_\mathbf{k}^\lambda$, we obtain

$$\int E^2\, d\mathbf{r} = 16\pi^3 \sum_\lambda \int d\mathbf{k}(\bar{a}_\mathbf{k}^\lambda a_\mathbf{k}^\lambda) + 8\pi^3 \int d\mathbf{k}\{(a_\mathbf{k}^1 a_{-\mathbf{k}}^1 - a_\mathbf{k}^2 a_{-\mathbf{k}}^2) e^{-2i\omega t}$$

$$+ (\bar{a}_\mathbf{k}^1 \bar{a}_{-\mathbf{k}}^1 - \bar{a}_\mathbf{k}^2 \bar{a}_{-\mathbf{k}}^2)\, e^{+2i\omega t}\} \tag{5.12}$$

Since from (3.14) and (3.15) **H** can be obtained from **E** by replacing $a_\mathbf{k}^\lambda$ by $b_\mathbf{k}^\lambda$, we have

$$\int H^2\, d\mathbf{r} = 16\pi^3 \sum_\lambda \int \mathrm{d}\mathbf{k}(\bar{b}_\mathbf{k}^\lambda b_\mathbf{k}^\lambda) + 8\pi^3 \int d\mathbf{k}\{(b_\mathbf{k}^1 b_{-\mathbf{k}}^1 - b_\mathbf{k}^2 b_{-\mathbf{k}}^2)\, e^{-2i\omega t}$$

$$+ (\bar{b}_\mathbf{k}^1 \bar{b}_{-\mathbf{k}}^1 - \bar{b}_\mathbf{k}^2 \bar{b}_{-\mathbf{k}}^2)\, e^{+2i\omega t}\}$$

Using (3.16) we get

$$\int H^2\, d\mathbf{r} = 16\pi^3 \sum_\lambda \int d\mathbf{k}(\bar{a}_\mathbf{k}^\lambda a_\mathbf{k}^\lambda) + 8\pi^3 \int d\mathbf{k}\{(a_\mathbf{k}^2 a_{-\mathbf{k}}^2 - a_\mathbf{k}^1 a_{-\mathbf{k}}^1)\, e^{-2i\omega t}$$

$$+ (\bar{a}_\mathbf{k}^2 \bar{a}_{-\mathbf{k}}^2 - \bar{a}_\mathbf{k}^1 \bar{a}_{-\mathbf{k}}^1)\, e^{2i\omega t} \tag{5.13}$$

Hence, on adding (5.12) and (5.13), the oscillatory terms cancel, and (5.10) gives

$$E = 4\pi^2 \sum_\lambda \int d\mathbf{k}\, (\bar{a}_\mathbf{k}^\lambda a_\mathbf{k}^\lambda) \tag{5.14}$$

Now for the momentum, we take the vector product of (3.14) and (3.15) after making the substitutions

$$\alpha_\mathbf{k}^\lambda = a_\mathbf{k}^\lambda e^{-i\omega t}$$

$$\beta_\mathbf{k}^\lambda = b_\mathbf{k}^\lambda e^{-i\omega t}$$

We get

$$\begin{aligned}\mathbf{E} \times \mathbf{H} = \sum_\lambda \sum_{\lambda'} \int d\mathbf{k} \int d\mathbf{k}' \, [\boldsymbol{\varepsilon}_\mathbf{k}^\lambda \times \boldsymbol{\varepsilon}_{\mathbf{k}'}^{\lambda'}]\{&\alpha_\mathbf{k}^\lambda \beta_{\mathbf{k}'}^{\lambda'} e^{i(\mathbf{k}+\mathbf{k}')\mathbf{r}} \\ &+ \bar{\alpha}_\mathbf{k}^\lambda \beta_{\mathbf{k}'}^{\lambda'} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} + \alpha_\mathbf{k}^\lambda \bar{\beta}_{\mathbf{k}'}^{\lambda'} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} \\ &+ \bar{\alpha}_\mathbf{k}^\lambda \bar{\beta}_{\mathbf{k}'}^{\lambda'} e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}\}\end{aligned}$$

$$\begin{aligned}\int \mathbf{E} \times \mathbf{H}\, d\mathbf{r} = 8\pi^3 \sum_\lambda \sum_{\lambda'} \int d\mathbf{k}\, \{&\boldsymbol{\varepsilon}_\mathbf{k}^\lambda \times \boldsymbol{\varepsilon}_\mathbf{k}^{\lambda'}(\bar{\alpha}_\mathbf{k}^\lambda \beta_\mathbf{k}^{\lambda'} + \alpha_\mathbf{k}^\lambda \bar{\beta}_\mathbf{k}^{\lambda'}) \\ &+ \boldsymbol{\varepsilon}_\mathbf{k}^\lambda \times \boldsymbol{\varepsilon}_{-\mathbf{k}}^{\lambda'}(\alpha_\mathbf{k}^\lambda \beta_{-\mathbf{k}}^{\lambda'} + \bar{\alpha}_\mathbf{k}^\lambda \bar{\beta}_{-\mathbf{k}}^{\lambda'})\}\end{aligned}$$

We have

$$\boldsymbol{\varepsilon}_\mathbf{k}^1 \times \boldsymbol{\varepsilon}_\mathbf{k}^2 = -\boldsymbol{\varepsilon}_\mathbf{k}^2 \times \boldsymbol{\varepsilon}_\mathbf{k}^1 = \frac{\mathbf{k}}{k}$$

$$\boldsymbol{\varepsilon}_\mathbf{k}^1 \times \boldsymbol{\varepsilon}_{-\mathbf{k}}^2 = \boldsymbol{\varepsilon}_\mathbf{k}^2 \times \boldsymbol{\varepsilon}_{-\mathbf{k}}^1 = -\frac{\mathbf{k}}{k}$$

and the other products vanish. Hence

$$\begin{aligned}\int \mathbf{E} \times \mathbf{H}\, d\mathbf{r} &= 8\pi^3 \int d\mathbf{k} \frac{\mathbf{k}}{k} \{(\bar{a}_\mathbf{k}^1 b_\mathbf{k}^2 + a_\mathbf{k}^1 \bar{b}_\mathbf{k}^2 - \bar{a}_\mathbf{k}^2 b_\mathbf{k}^1 - a_\mathbf{k}^2 \bar{b}_\mathbf{k}^1) \\ &\qquad - (a_\mathbf{k}^1 b_{-\mathbf{k}}^2 + a_\mathbf{k}^2 b_{-\mathbf{k}}^1)\, e^{-2i\omega t} \\ &\qquad - (\bar{a}_\mathbf{k}^1 \bar{b}_{-\mathbf{k}}^2 + \bar{a}_\mathbf{k}^2 b_{-\mathbf{k}}^1)\, e^{2i\omega t}\} \\ &= 8\pi^3 \int d\mathbf{k} \frac{\mathbf{k}}{k} \{2(\bar{a}_\mathbf{k}^1 a_\mathbf{k}^1 + a_\mathbf{k}^2 \bar{a}_\mathbf{k}^2) \\ &\qquad - (a_\mathbf{k}^1 a_{-\mathbf{k}}^1 - a_\mathbf{k}^2 a_{-\mathbf{k}}^2)\, e^{-2i\omega t} \\ &\qquad - (\bar{a}_\mathbf{k}^1 \bar{a}_{-\mathbf{k}}^1 - \bar{a}_\mathbf{k}^2 \bar{a}_{-\mathbf{k}}^2)\, e^{2i\omega t}\}\end{aligned}$$

The integral of the oscillatory terms vanishes since this part of the integrand changes sign when $\mathbf{k}$ is replaced by $-\mathbf{k}$. Hence (5.11) becomes

$$\mathbf{p} = \frac{4\pi^2}{c} \sum_{\lambda} \int d\mathbf{k} \, \frac{\mathbf{k}}{k} (\bar{a}_{\mathbf{k}}^{\lambda} a_{\mathbf{k}}^{\lambda}) \tag{5.15}$$

6 RADIATION FROM A CHARGE AND CURRENT DISTRIBUTION

We shall first give a method of calculation which is most convenient when only the rate of radiation of energy is required. We write $\mathbf{E}$ as the sum of "transverse" and "longitudinal" fields $\mathbf{E}_{\perp}$ and $\mathbf{E}_{\parallel}$ respectively, which are defined as follows:

$$\nabla \cdot \mathbf{E}_{\perp} = 0$$

$$\nabla \times \mathbf{E}_{\parallel} = 0$$

In other words, we break up $\mathbf{E}$ into its solenoidal and irrotational parts. Maxwell's Eqs. (2.3) and (2.1) then become

$$\nabla \cdot \mathbf{E}_{\parallel} = 4\pi\varrho$$

$$\nabla \times \mathbf{E}_{\perp} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$

We can write

$$\mathbf{E}_{\parallel} = -\nabla\varphi$$

$$\Delta\varphi = -4\pi\varrho$$

and

$$\varphi(\mathbf{r}, t) = \int \frac{\varrho(\mathbf{r}', t)\, d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \tag{6.1}$$

$$\mathbf{E}_{\parallel} = -\nabla \int \frac{\varrho(\mathbf{r}', t)\, d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

$\mathbf{E}_{\parallel}$ is calculated by keeping t constant. Maxwell's equations can now be written in the form

$$\nabla \cdot \mathbf{E}_{\perp} = 0 \tag{6.2}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{6.3}$$

$$\nabla \times \mathbf{E}_{\perp} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = 0 \tag{6.4}$$

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}_{\perp}}{\partial t} = 4\pi\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}_{\parallel}}{\partial t} \tag{6.5}$$

Now

$$\frac{1}{c}\frac{\partial \mathbf{E}_{\parallel}}{\partial t} = -\nabla \frac{1}{c}\frac{\partial}{\partial t}\int \frac{\varrho(\mathbf{r}', t)\, d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

and using the equation of continuity (2.5) we obtain

$$\frac{1}{c}\frac{\partial \mathbf{E}_{\parallel}}{\partial t} = \nabla \int \frac{\nabla' \cdot \mathbf{j}(\mathbf{r}', t)\, d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

Thus

$$\nabla \cdot \left(4\pi\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}_{\parallel}}{\partial t}\right) = 4\pi\, \nabla \cdot \mathbf{j} + \int \nabla' \cdot \mathbf{j}(\mathbf{r}', t)\, d\mathbf{r}'\, \Delta \frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

and since

$$\Delta \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\, \delta(\mathbf{r}-\mathbf{r}')$$

it follows that,

$$\nabla \cdot \left(4\pi\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}_{\parallel}}{\partial t}\right) = 0$$

Thus $\mathbf{E}_{\parallel}$ takes out the longitudinal part from $\mathbf{j}$. Hence $\mathbf{E}_{\perp}$ and $\mathbf{H}$ can be represented in terms of transverse waves. We shall write

$$4\pi\mathbf{j}_{\perp} = 4\pi\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}_{\parallel}}{\partial t}$$

It means that only the component of $\mathbf{j}$ perpendicular to the propagation vector excites transverse electric waves. For if we make a Fourier expansion of $\mathbf{j}$,

$$\mathbf{j}(\mathbf{r}, t) = \sum_{\lambda=1,2,3} \int d\mathbf{k}\, \boldsymbol{\varepsilon}_{\mathbf{k}}^{\lambda}\{g_{\lambda}(\mathbf{k}, t)\, e^{i\mathbf{k}\cdot\mathbf{r}} + \bar{g}_{\lambda}(\mathbf{k}, t)\, e^{-i\mathbf{k}\cdot\mathbf{r}}\}$$

where

$$\boldsymbol{\varepsilon}_{\mathbf{k}}^{3} = \frac{\mathbf{k}}{k}$$

and

$$g_{\lambda}(\mathbf{k}, t) = \frac{1}{8\pi^3}\int d\mathbf{r}\, \mathbf{j}(\mathbf{r}, t) \cdot \boldsymbol{\varepsilon}_{\mathbf{k}}^{\lambda}\, e^{-i\mathbf{k}\cdot\mathbf{r}}$$

then

$$\nabla \cdot \mathbf{j} = i \sum_{\lambda} \int d\mathbf{k}\, \boldsymbol{\varepsilon}_{\mathbf{k}}^{\lambda} \cdot \mathbf{k}\{g_{\lambda}(\mathbf{k}, t)\, e^{i\mathbf{k}\cdot\mathbf{r}} - \bar{g}_{\lambda}(\mathbf{k}, t)\, e^{-i\mathbf{k}\cdot\mathbf{r}}\}$$

$$= i \int d\mathbf{k}\, k\{g_3(\mathbf{k}, t)\, e^{i\mathbf{k}\cdot\mathbf{r}} - \bar{g}_3(\mathbf{k}, t)\, e^{-i\mathbf{k}\cdot\mathbf{r}}\}$$

Thus the expansion of $\mathbf{j}_\perp$ is

$$\mathbf{j}_\perp(\mathbf{r}, t) = \sum_{\lambda=1,2} \int d\mathbf{k}\, \boldsymbol{\varepsilon}_\mathbf{k}^\lambda \{g_\lambda(\mathbf{k}, t)\, e^{i\mathbf{k}\cdot\mathbf{r}} + \bar{g}_\lambda(\mathbf{k} \cdot t)\, e^{-i\mathbf{k}\cdot\mathbf{r}}\} \tag{6.6}$$

This follows from the fact that, since $\nabla \cdot \mathbf{j}_\perp = 0$, $\mathbf{j}_\perp$ cannot have any component parallel to $\mathbf{k}$.

We now introduce a vector potential $\mathbf{A}(\mathbf{r}, t)$, which we define in this case by the following equations:

$$\mathbf{H} = \nabla \times \mathbf{A}$$

$$\mathbf{E}_\perp = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\nabla \cdot \mathbf{A} = 0$$

Then Eqs. (6.2), (6.3) and (6.4) are automatically satisfied, and we are left with just (6.5), which becomes

$$-\Delta \mathbf{A} + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 4\pi \mathbf{j}_\perp \tag{6.7}$$

Let us now make a Fourier expansion of $\mathbf{A}$. Since $\nabla \cdot \mathbf{A} = 0$, $\mathbf{A}$ has no component parallel to $\mathbf{k}$. Hence

$$\mathbf{A} = \sum_{\lambda=1,2} \int d\mathbf{k}\, \boldsymbol{\varepsilon}_\mathbf{k}^\lambda \{\eta_\lambda(\mathbf{k}, t)\, e^{i\mathbf{k}\cdot\mathbf{r}} + \bar{\eta}_\lambda(\mathbf{k}, t)\, e^{-i\mathbf{k}\cdot\mathbf{r}}\} \tag{6.8}$$

where

$$\eta_\lambda(\mathbf{k}, t) = \frac{1}{8\pi^3} \int d\mathbf{r}\, \mathbf{A}(\mathbf{r}, t) \cdot \boldsymbol{\varepsilon}_\mathbf{k}^\lambda\, e^{-i\mathbf{k}\cdot\mathbf{r}}$$

Substituting this expression for $\mathbf{A}$ into (6.7) and using (6.6) we find

$$\sum_{\lambda=1,2} \int d\mathbf{k}\, \boldsymbol{\varepsilon}_\mathbf{k}^\lambda \Bigg[\left\{k^2\eta_\lambda + \frac{1}{c^2}\ddot{\eta}_\lambda - 4\pi g_\lambda\right\} e^{i\mathbf{k}\cdot\mathbf{r}}$$
$$+ \left\{k^2\bar{\eta}_\lambda + \frac{1}{c^2}\ddot{\eta}_\lambda - 4\pi g_\lambda\right\} e^{-i\mathbf{k}\cdot\mathbf{r}}\Bigg] = 0$$

where a dot denotes partial differentiation with respect to t. Hence

$$\ddot{\eta}_\lambda + c^2k^2\eta_\lambda = 4\pi c^2 g_\lambda$$

This is the differential equation of forced oscillation. Its complementary solution, that is, the free vibration, is given by

$$\eta_\lambda = L_\lambda(\mathbf{k})\, e^{i\omega t} + M_\lambda(\mathbf{k})\, e^{-i\omega t}$$

Putting this into (6.8) and calculating $\mathbf{E}_\perp$ we get

$$\mathbf{E}_\perp = -\frac{i\omega}{c}\sum_{\lambda=1,2}\int d\mathbf{k}\, \boldsymbol{\varepsilon}_{\mathbf{k}}^{\lambda}\,[(L_\lambda e^{i\omega t} - M_\lambda e^{-i\omega t})\, e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$- (\bar{L}_\lambda e^{-i\omega t} - \bar{M}_\lambda e^{i\omega t})\, e^{-i\mathbf{k}\cdot\mathbf{r}}] \tag{6.9}$$

In free space $\mathbf{E}_\perp = \mathbf{E}$, and comparing (6.9) with (3.14) we find

$$L_\lambda = 0$$

$$\frac{i\omega}{c} M_\lambda = +a_{\mathbf{k}}^{\lambda}$$

and hence

$$\eta_\lambda = -\frac{ic}{\omega}\, a_{\mathbf{k}}^{\lambda} e^{-i\omega t}$$

If we consider the current to vary harmonically with the time and write

$$g_\lambda = \gamma_\lambda e^{-i\nu t}$$

then the particular integral is

$$\frac{4\pi c^2 \gamma_\lambda}{\omega^2 - \nu^2}\, e^{-i\nu t}$$

and hence the general solution is

$$\eta_\lambda = L_\lambda e^{i\omega t} + M_\lambda e^{-i\omega t} + \frac{4\pi c^2 \gamma_\lambda}{\omega^2 - \nu^2}\, e^{-i\nu t}$$

Let us choose the constants so that $\mathbf{E}_\perp = 0$ and $\mathbf{H} = 0$ at $t = 0$. Then

$$L_\lambda - M_\lambda - \frac{4\pi c^2 \gamma_\lambda}{\omega^2 - \nu^2}\frac{\nu}{\omega} = 0$$

$$L_\lambda + M_\lambda + \frac{4\pi c^2 \gamma_\lambda}{\omega^2 - \nu^2} = 0$$

from which we find

$$L_\lambda = -\frac{2\pi c^2 \gamma_\lambda}{\omega(\omega + \nu)}$$

$$M_\lambda = -\frac{2\pi c^2 \gamma_\lambda}{\omega(\omega - \nu)}$$

$$\eta_\lambda = 4\pi c^2 \gamma_\lambda \left\{ \frac{e^{-i\nu t}}{\omega^2 - \nu^2} - \frac{e^{i\omega t}}{2\omega(\omega + \nu)} - \frac{e^{-i\omega t}}{2\omega(\omega - \nu)} \right\} \tag{6.10}$$

and for $\omega \approx \nu$

$$\eta_\lambda \approx -\frac{2\pi c^2 \gamma_\lambda e^{-i\omega t}}{\omega(\omega-\nu)}\{1 - e^{+i(\omega-\nu)t}\} \tag{6.11}$$

The rate of radiation of energy is obtained by finding the part of

$$E = \frac{1}{8\pi}\int (E^2 + H^2)\, d\mathbf{r}$$
$$= \frac{1}{8\pi}\int (E_{\|}^2 + E_{\perp}^2 + H^2)\, d\mathbf{r}$$

which increases secularly with t. It is clear that $\mathbf{E}_{\|}$ does not contribute to this energy since it is just the coulomb field of the charges. Thus we need only consider the energy in the radiation field

$$E_{rad} = \frac{1}{8\pi}\int (E_{\perp}^2 + H^2)\, d\mathbf{r}$$

From (6.8) and the definition of **A** we have

$$\mathbf{E}_{\perp} = \sum_{\lambda=1,2}\int d\mathbf{k}\, \boldsymbol{\varepsilon}_{\mathbf{k}}^{\lambda}\left\{-\frac{1}{c}\dot{\eta}_\lambda e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{1}{c}\dot{\bar{\eta}}_\lambda e^{-i\mathbf{k}\cdot\mathbf{r}}\right\}$$

$$\mathbf{H} = \sum_{\lambda=1,2}\int d\mathbf{k}\, \mathbf{n} \times \boldsymbol{\varepsilon}_{\mathbf{k}}^{\lambda}\left\{ik\eta_\lambda e^{i\mathbf{k}\cdot\mathbf{r}} - ik\bar{\eta}_\lambda e^{-i\mathbf{k}\cdot\mathbf{r}}\right\}, \quad \mathbf{n} = \frac{\mathbf{k}}{k}.$$

Now the dominant contributions to these integrals come from the region $\omega \approx \nu$, where we have

$$\dot{\eta}_\lambda \approx -i\omega\eta_\lambda$$

so that, comparing these equations with (3.14) and (3.15), we can make the correspondences

$$a_{\mathbf{k}}^{\lambda}\, e^{-i\omega t} = \frac{i\omega}{c}\eta_\lambda$$

$$b_{\mathbf{k}}^{1}\, e^{-i\omega t} = -ik\eta_2$$

$$b_{\mathbf{k}}^{2}\, e^{-i\omega t} = ik\eta_1$$

which satisfy the relations (3.16). Hence we can obtain the value of E in terms of $\eta_{\mathbf{k}}^{\lambda}$ by making these correspondences in (5.14). The result is

$$E_{rad} = 4\pi^2 \sum_{\lambda=1,2}\int d\mathbf{k}\left(\frac{\omega^2}{c^2}\bar{\eta}_\lambda\eta_\lambda\right)$$

From (6.11) we have

$$\bar{\eta}_\lambda \eta_\lambda = \frac{8\pi^2 c^4 \bar{\gamma}_\lambda \gamma_\lambda}{\omega^2(\omega - \nu)^2}\{1 - \cos(\omega - \nu)\, t\}$$

Thus

$$E_{rad} = 32\pi^4 c^2 \sum_{\lambda=1,2} \int\int k^2\, dk\, d\Omega\, \bar{\gamma}_\lambda \gamma_\lambda \frac{\{1 - \cos(\omega - \nu)\, t\}}{(\omega - \nu)^2}$$

Since the integrand has a sharp maximum for $\omega = \nu$, we can take $k^2\bar{\gamma}_\lambda\gamma_\lambda$ outside the k integral, which then becomes

$$\int_0^\infty \frac{d\omega}{c} \frac{\{1 - \cos(\omega - \nu)\, t\}}{(\omega - \nu)^2} = \frac{t}{c} \int_{-\nu t/2}^{\infty} \frac{\sin^2 x}{x^2}\, dx$$

Since the integrand falls off very rapidly, for $t \gg 1/\nu$, we can take the lower limit as $-\infty$. Therefore

$$\text{integral} = \frac{\pi t}{c}$$

and

$$E_{rad} = \frac{32\pi^5 \nu^2 t}{c} \sum_{\lambda=1,2} \int d\Omega\, \bar{\gamma}_\lambda \gamma_\lambda$$

Hence the rate of radiation of energy in the solid angle $d\Omega$ is

$$\begin{aligned} dR &= d\Omega \frac{32\pi^5 \nu^2}{c}\{|\gamma_1|^2 + |\gamma_2|^2\} \\ &= d\Omega \frac{\nu^2}{2\pi c} \sum_{\lambda=1,2} \left| \int d\mathbf{r}\, \mathbf{j}(\mathbf{r}, t) \cdot \boldsymbol{\varepsilon}_\nu^\lambda e^{-i\frac{\nu}{c}\mathbf{n}\cdot\mathbf{r}} \right|^2 \\ &= d\Omega \frac{\nu^2}{2\pi c} \left| \int d\mathbf{r}\, [\mathbf{j} \times \mathbf{n}]\, e^{-i\frac{\nu}{c}\mathbf{n}\cdot\mathbf{r}} \right|^2 \end{aligned} \qquad (6.12)$$

EXERCISE 9 Show that for the case of zero total charge

$$W_{\parallel} \equiv \frac{1}{8\pi} E_{\parallel}^2 \propto \frac{1}{r^6}$$

$$W_{\perp} \equiv \frac{1}{8\pi}(E_{\perp}^2 + H^2) \propto \frac{1}{r^2}$$

and hence

$$\frac{W_{\parallel}}{W_{\perp}} \to 0 \quad \text{as} \quad r \to \infty$$

Also give an idea of the distance where $W_{\parallel}$ becomes negligible.

Let us apply (6.12) to calculate the radiation from an antenna. Consider a current in a conductor whose breadth is very small compared to its length L. Let the z axis be along the conductor. Then only the z-component of $\mathbf{j}$ is different from zero; suppose it to be of the form

$$i = \int\int j_z \, dx \, dy = \frac{I}{2}\left(\sin\frac{\pi z}{L}\right)(e^{i\nu t} + e^{-i\nu t})$$

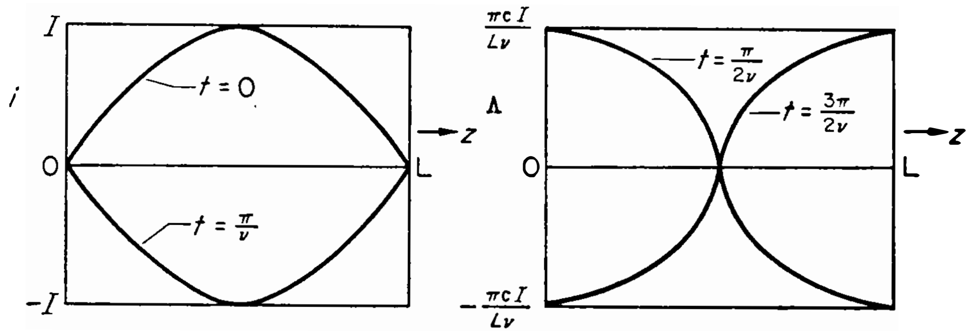
Now

$$\frac{\partial \varrho}{\partial t} = -c \, \nabla \cdot \mathbf{j}$$

so that

$$\Lambda = \int\int \varrho \, dx \, dy = \Lambda_0(z) - \frac{\pi c I}{L\nu} \cos\frac{\pi z}{L} \sin \nu t$$

Hence the current and charge distribution in the wire have the following forms.



It is clear that Λ_0 cannot be a function of z for a uniform, straight isolated wire in free space, and it may be taken as zero.

If $\lambda = \dfrac{2\pi c}{\nu} \gg L$, then the integral

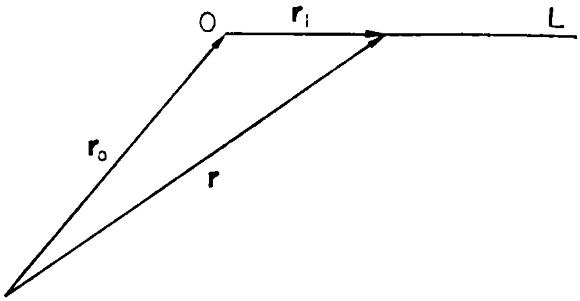
$$\int d\mathbf{r} \, [\mathbf{j} \times \mathbf{n}] \, e^{-\frac{2\pi i}{\lambda}\mathbf{n}\cdot\mathbf{r}}$$

is easy to perform. Put $\mathbf{r} = \mathbf{r}_0 + \mathbf{r}_1$. Then the integral becomes

$$e^{-\frac{2\pi i}{\lambda}\mathbf{n}\cdot\mathbf{r}_0} \int d\mathbf{r}_1 \, [\mathbf{j}(\mathbf{r}_1, t) \times \mathbf{n}] \, e^{-\frac{2\pi i}{\lambda}(\mathbf{n}\cdot\mathbf{r}_1)}$$

and under our assumption we have

$$\frac{|\mathbf{r}_1|}{\lambda} \ll 1 \quad \text{so that} \quad e^{-\frac{2\pi i}{\lambda}(\mathbf{n}\cdot\mathbf{r}_1)} \approx 1$$



and hence

$$dR = d\Omega \frac{\nu^2}{2\pi c} \left(\frac{LI}{\pi}\right)^2 \sin^2\theta$$

$$= d\Omega \frac{\nu^2 L^2 I^2}{2\pi^3 c} \sin^2\theta$$

Here θ is the angle between the z-axis and $\mathbf{n}$.

EXERCISE 10 Taking $R = 10$ watts, $\nu = 10^7$ sec^{-1}, and $L = 10$ cm, find the maximum charge accumulated on one-half of the wire.

For the general case we have to evaluate the integral

$$I = \int_0^L dz \sin\frac{\pi z}{L}\, e^{-\frac{2\pi i}{\lambda} z\cos\theta}$$

The integration can be carried out by writing the exponential factor as a sum of cosine and sine terms; the resulting trigonometric integrals can be evaluated by elementary means. The result is

$$I = \frac{L}{\pi\left(1 - \dfrac{4L^2}{\lambda^2}\cos^2\theta\right)}\left(1 + e^{-\frac{2\pi i}{\lambda}L\cos\theta}\right)$$

Hence

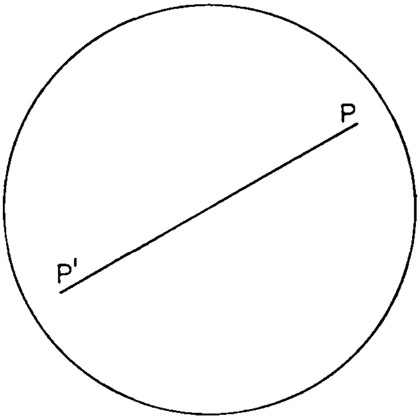
$$|I|^2 = \frac{4}{\pi^2}\,\frac{L^2}{\left(1 - \dfrac{4L^2}{\lambda^2}\cos^2\theta\right)^2}\cos^2\frac{\pi L\cos\theta}{\lambda}$$

and

$$dR = d\Omega\frac{\nu^2 I^2 L^2}{2\pi^3 c}\,\frac{\sin^2\theta}{\left(1 - \dfrac{4L^2}{\lambda^2}\cos^2\theta\right)^2}\cos^2\frac{\pi L\cos\theta}{\lambda}$$

We note that when $\mathbf{n}$ is perpendicular to $\mathbf{r}_1 (\sin\theta = 1)$ this result gives the same value as the approximate expression obtained above for $\lambda \gg L$.

Consider a spherically symmetric charge distribution which oscillates radially. Such an oscillation of charge cannot give rise to any radiation, for we have seen that only the part of the current perpendicular to the propagation vector gives any radiation, and if we consider any plane through the charge distribution, to any point P, then on the plane there is a point P', symmetrically placed with respect to it, where the value of the current is equal and opposite to that at P. Thus the integral (6.12) vanishes, and there is no radiation. Another proof can be given by using Gauss's theorem,



which states that the field outside a spherically symmetric distribution of charge is the same as that when all the charge is concentrated at the center. It follows from this that with our distribution of charge the field is radial and constant. Hence there is no radiation.

Let us now consider radiation from an atomic system. In nonrelativistic theory, the expression for the matrix element of the current between two stationary states m and n is

$$\mathbf{j}_{mn}(\mathbf{r}, t) = e^{\frac{i}{\hbar}(E_m - E_n)t}\, \bar{u}_m \left(\frac{\hbar}{m}\frac{e}{ic}\nabla\right) u_n$$

This is obtained from Schrödinger's equation and the condition of continuity. By analogy the rate of radiation from state n to m is

$$dR_{n \to m} = \frac{d\Omega}{2\pi c}\left(\frac{E_m - E_n}{\hbar}\right)^2 \frac{\hbar^2 e^2}{m^2 c^2}\left|\int d\mathbf{r}\, e^{\frac{i}{\hbar c}(E_n - E_m)\mathbf{n}\cdot\mathbf{r}}\, \bar{u}_m \mathbf{n} \times \nabla u_n\right|^2 \qquad (6.13)$$

Now

$$\frac{\hbar c}{E_n - E_m} = \lambda \sim \frac{7 \times 10^{-28} \times 3 \times 10^{10}}{8 \times 10^{-12}} \sim 10^{-4}\,\text{cm}$$

and the size of the atom is $d = 10^{-8}$ cm so that $d/\lambda \ll 1$, and we may neglect the exponential. We may make a rough estimate of (6.13) by setting

$$\left|\int d\mathbf{r}\,(\bar{u}_m \mathbf{n} \times \nabla u_n)\right|^2 \sim \frac{1}{d^2}$$

Introducing the partial line breadth Γ_{nm}, we have, for the transition rate,

$$\frac{\Gamma_{nm}}{\hbar} = \frac{R_{n\to m}}{E_n - E_m} \sim \frac{e^2}{\hbar c}\,\frac{E_n - E_m}{\hbar}\left(\frac{\hbar}{mcd}\right)^2$$

$$\sim \frac{1}{137}\,(\nu)\left(\frac{\hbar}{mc}\cdot\frac{me^2}{\hbar^2}\right)^2$$

where we have taken

$$d \sim a = \frac{\hbar^2}{me^2}$$

Therefore

$$\frac{\Gamma_{nm}}{\hbar} \sim \left(\frac{1}{137}\right)^3 \nu, \quad \Gamma_{nm} \sim \left(\frac{1}{137}\right)^3 (E_n - E_m)$$

Or again, since $\bar{p}^2 d^2 \sim \hbar^2$, $d^2 \sim \dfrac{\hbar^2}{m^2\bar{v}^2}$ we have

$$\left(\frac{\hbar}{mcd}\right)^2 \sim \frac{\bar{v}^2}{c^2}$$

$$\frac{\Gamma_{nm}}{\hbar} = \frac{1}{137}\,\nu\,\frac{\bar{v}^2}{c^2}$$

The natural lifetime is the reciprocal of this.

Exercise 11 Considering the dipole term only, that is, neglecting the exponential in the integrand, calculate the mean lifetime of the state u_n where there is only the state u_m below it, where

$$u_n = \frac{1}{\sqrt{2\pi a^3}}\,\frac{r}{4\pi a}\,e^{-2r/2a}\cos\theta$$

$$u_m = \frac{1}{\sqrt{\pi a^3}}\,e^{-r/a}$$

$$a = \frac{\hbar^2}{me^2}$$

Also show that the transition from $u_{n'}$ to u_m, where

$$u_{n'} = \frac{1}{\sqrt{2\pi a^3}}\left(\frac{1}{2} - \frac{r}{4a}\right) e^{-r/2a}$$

is completely forbidden.

Using Eq. (6.13), we can prove that a free electron does not radiate. The wave function of a free electron can be taken as

$$u = e^{i\mathbf{k}\cdot\mathbf{r}}$$

and the energy E_k corresponding to this state obtained from the relation

$$H\psi = E\psi$$

which for this case is

$$\frac{p^2}{2m} e^{i\mathbf{k}\cdot\mathbf{r}} = E_k e^{i\mathbf{k}\cdot\mathbf{r}}$$

Now $\quad \mathbf{p} = \frac{\hbar}{i}\nabla, \qquad p^2 = -\hbar^2\Delta, \qquad$ which yields

$$\frac{\hbar^2 k^2}{2m} e^{i\mathbf{k}\cdot\mathbf{r}} = E_k e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$E_k = \frac{\hbar^2 k^2}{2m}$$

We want to show that the integral

$$\int d\mathbf{r}\, e^{i\frac{E_m - E_n}{\hbar c}(\mathbf{n}\cdot\mathbf{r})} \bar{u}_m \mathbf{n} \times \nabla u_n$$

vanishes for this case. Putting in the corresponding values, the integral is

$$i\int d\mathbf{r}\, e^{i\frac{\hbar(k^2 - k'^2)}{2mc}\mathbf{n}\cdot\mathbf{r}} e^{-i\mathbf{k}'\cdot\mathbf{r}} \mathbf{n} \times \mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$= (2\pi)^3\, i\, \delta\left(\frac{\hbar(k^2 - k'^2)}{2mc}\mathbf{n} - \mathbf{k}' + \mathbf{k}\right)\mathbf{n} \times \mathbf{k}$$

Thus the integral is zero unless

$$\mathbf{k}' = \mathbf{k} + \frac{\hbar(k^2 - k'^2)}{2mc}\mathbf{n}$$

that is,

$$k'^2 = k^2 + \frac{\hbar(k^2 - k'^2)}{mc}\mathbf{n}\cdot\mathbf{k} + \frac{\hbar^2(k^2 - k'^2)^2}{4m^2c^2}$$

The values of $k^2 - k'^2$ which satisfy this relation are zero and

$$k^2 - k'^2 = -\frac{4m^2c^2}{\hbar^2}\left\{1 + \frac{\hbar\mathbf{n}\cdot\mathbf{k}}{mc}\right\}$$

Now

$$\left|\frac{\hbar\mathbf{n}\cdot\mathbf{k}}{mc}\right| \leqq \frac{mv}{mc} \leqq 1$$

and since $k^2 - k'^2$ cannot be negative, the second solution is not good. Hence $k' = k$, and there is no radiation.

7 SOLUTION OF MAXWELL'S EQUATIONS IN TERMS OF RETARDED POTENTIALS

We introduce the potentials φ and $\boldsymbol{\psi}$ defined by the following equations:

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial\boldsymbol{\psi}}{\partial t} \tag{7.1}$$

$$\mathbf{H} = \nabla\times\boldsymbol{\psi} \tag{7.2}$$

$$\nabla\cdot\boldsymbol{\psi} + \frac{1}{c}\frac{\partial\varphi}{\partial t} = 0 \tag{7.3}$$

(7.3) is the Lorentz condition which restricts the gauge. That is, (7.1) and (7.2) do not determine φ and $\boldsymbol{\psi}$ completely, for if

$$\left.\begin{aligned} \varphi' &= \varphi - \frac{1}{c}\frac{\partial\Lambda}{\partial t} \\ \boldsymbol{\psi}' &= \boldsymbol{\psi} + \nabla\Lambda \end{aligned}\right\} \tag{7.4}$$

where Λ is any scalar point function, then φ' and $\boldsymbol{\psi}'$ give the same field as φ and $\boldsymbol{\psi}$. Thus there is a family of transformations generated by Λ which give the same field. Suppose

$$\frac{1}{c}\frac{\partial\varphi'}{\partial t} + \nabla\cdot\boldsymbol{\psi}' = \gamma$$

Then putting in (7.4) we see that φ and ψ satisfy the Lorentz condition if

$$\Box\Lambda = -\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} + \Delta\Lambda = +\gamma$$

Now with the above definition of φ and $\boldsymbol{\psi}$ we see that Maxwell's Eqs. (2.2) and (2.3) are automatically satisfied, and (2.1) and (2.4) give

$$-\Delta\varphi - \frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot\boldsymbol{\psi} = 4\pi\varrho$$

$$\Box\,\varphi = -4\pi\varrho \tag{7.5}$$

and

$$\nabla\times(\nabla\times\boldsymbol{\psi}) + \frac{1}{c}\frac{\partial}{\partial t}\nabla\varphi + \frac{1}{c^2}\frac{\partial^2\boldsymbol{\psi}}{\partial t^2} = 4\pi\mathbf{j}$$

$$-\Delta\boldsymbol{\psi} + \nabla\left\{\nabla\cdot\boldsymbol{\psi} + \frac{1}{c}\frac{\partial\varphi}{\partial t}\right\} + \frac{1}{c^2}\frac{\partial^2\boldsymbol{\psi}}{\partial t^2} = 4\pi\mathbf{j}$$

$$\Box\,\boldsymbol{\psi} = -4\pi\mathbf{j} \tag{7.6}$$

Thus we have to solve Eqs. (7.5) and (7.6) with the condition (7.3). These equations are consistent since

$$\Box\left\{\frac{1}{c}\frac{\partial\varphi}{\partial t} + \nabla\cdot\boldsymbol{\psi}\right\} = -4\pi\left\{\frac{1}{c}\frac{\partial\varrho}{\partial t} + \nabla\cdot\mathbf{j}\right\} = 0$$

We shall show that the solutions of (7.5) are

$$\varphi_{\mp}(\mathbf{r}, t) = \int d\mathbf{r}'\,\frac{\varrho\left(\mathbf{r}', t \mp \dfrac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}$$

where φ_- and φ_+ are called the retarded and advanced potentials respectively. The general solution is

$$\varphi = a\varphi_+ + (1-a)\,\varphi_-$$

where a is any real number. Similarly, the solutions of (7.6) are

$$\boldsymbol{\psi}_{\mp}(\mathbf{r}, t) = \int d\mathbf{r}'\,\frac{j\left(\mathbf{r}', t \mp \dfrac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}$$

To derive these solutions, make the Fourier expansions

$$\varrho(\mathbf{r}, t) = \int_{-\infty}^{+\infty} \varrho_\nu(\mathbf{r})\,e^{i\nu t}\,d\nu$$

$$\varrho_\nu(\mathbf{r}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varrho(\mathbf{r}, t)\,e^{-i\nu t}\,dt$$

$$\varphi(\mathbf{r}, t) = \int_{-\infty}^{+\infty} \varphi_\nu(\mathbf{r})\, e^{i\nu t}\, d\nu$$

$$\varphi_\nu(\mathbf{r}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi(\mathbf{r}, t)\, e^{-i\nu t}\, dt$$

Substituting these values in (7.5) we have

$$\int_{-\infty}^{+\infty} \left\{ \Delta\varphi_\nu + \frac{\nu^2}{c^2}\varphi_\nu + 4\pi\varrho_\nu \right\} e^{i\nu t}\, d\nu = 0$$

$$\Delta\varphi_\nu + \frac{\nu^2}{c^2}\varphi_\nu = -4\pi\varrho_\nu$$

We shall solve this equation by using a Green's function, $G(\mathbf{x}, \mathbf{x}')$. It has the property that if

$$D\varphi = f$$

where D is an operator, which in our case is $\Delta + \dfrac{\nu^2}{c^2}$, then

$$\varphi(\mathbf{x}) = \int f(\mathbf{x}')\, G(\mathbf{x}, \mathbf{x}')\, d\mathbf{x}'$$

Applying the operator D to this equation, we find

$$DG(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$$

For our case, there are two different G's

$$G\pm = -\frac{e^{\pm \frac{i\nu}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi\, |\mathbf{r} - \mathbf{r}'|}$$

Hence for large values of r, $\varphi_\nu e^{i\nu t}$ behaves like

$$\frac{e^{\pm i \frac{\nu}{c} r + i\nu t}}{r}$$

so that the + sign gives incoming waves and the − sign gives outgoing waves. In calculating the radiation from a charge, only the − sign should be used. It is interesting to note that we can make an oscillating charge stop radiating by sending in radiation of suitable amplitude and phase.

To verify the formula for the Green's function, take $\mathbf{r}' = 0$. Then

$$G = -\frac{e^{-i\frac{\nu}{c}r}}{4\pi r}$$

$$\Delta G = \delta(\mathbf{r})\, e^{-i\frac{\nu}{c}r} - \frac{1}{4\pi r}\left(\frac{\partial^2}{\partial r^2} e^{-i\frac{\nu}{c}r} + \frac{2}{r}\frac{\partial}{\partial r} e^{-i\frac{\nu}{c}r}\right)$$

$$- \frac{2}{4\pi}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right)\left(\frac{\partial}{\partial r} e^{-i\frac{\nu}{c}r}\right)$$

$$= \delta(\mathbf{r})\, e^{-i\frac{\nu}{c}r} + \frac{\nu^2}{c^2}\frac{1}{4\pi r} e^{-i\frac{\nu}{c}r}$$

whence

$$\left(\Delta + \frac{\nu^2}{c^2}\right) G = \delta(\mathbf{r})$$

The exponential coefficient of $\delta(\mathbf{r})$ is left out since $\delta(\mathbf{r})$ vanishes everywhere except at $\mathbf{r} = 0$, and here the exponential is unity.

Thus we have

$$\varphi_\nu = \int d\mathbf{r}\, \varrho_\nu(\mathbf{r}') \frac{e^{-i\frac{\nu}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|}$$

$$\varphi = \int_{-\infty}^{+\infty} d\nu\, e^{i\nu t} \int d\mathbf{r}'\, \varrho_\nu(\mathbf{r}') \frac{e^{-i\frac{\nu}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|}$$

$$= \int_{-\infty}^{+\infty} d\nu\, e^{i\nu t} \int d\mathbf{r}' \frac{e^{-i\frac{\nu}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varrho(\mathbf{r}', t')\, e^{-i\nu t'}\, dt'$$

$$= \int d\mathbf{r}' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \int_{-\infty}^{+\infty} dt'\, \varrho(\mathbf{r}', t')\, \delta\!\left(t - t' - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)$$

$$= \int d\mathbf{r}' \frac{\varrho\left(\mathbf{r}', t - \dfrac{|\mathbf{r} - \mathbf{r}'|}{c}\right)}{|\mathbf{r} - \mathbf{r}'|} \tag{7.7}$$

It can be shown in exactly the same way that

$$\psi = \int d\mathbf{r}' \frac{\mathbf{j}\left(\mathbf{r}', t - \dfrac{|\mathbf{r} - \mathbf{r}'|}{c}\right)}{|\mathbf{r} - \mathbf{r}'|} \tag{7.8}$$

It is not difficult to show that these solutions satisfy the Lorentz condition. Setting

$$\tau = \frac{|\mathbf{r} - \mathbf{r}'|}{c}$$

we have

$$\frac{1}{c}\frac{\partial \varphi}{\partial t} = \int \frac{d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}\frac{1}{c}\frac{\partial}{\partial t}\varrho(\mathbf{r}', t - \tau)$$

Also

$$\begin{aligned}
\frac{\partial \psi_x}{\partial x} &= \int d\mathbf{r}' \left\{\frac{\partial}{\partial x}\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right\} j_x(\mathbf{r}', t - \tau) \\
&\quad + \int d\mathbf{r}' \frac{1}{|\mathbf{r} - \mathbf{r}'|}\frac{\partial}{\partial \tau} j_x(\mathbf{r}', t - \tau)\frac{\partial \tau}{\partial x} \\
&= -\int d\mathbf{r}' \left\{\frac{\partial}{\partial x'}\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right\} j_x(\mathbf{r}', t - \tau) \\
&\quad - \int \frac{d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}\frac{\partial}{\partial \tau}(j_x(\mathbf{r}', t - \tau))\frac{\partial \tau}{\partial x'}
\end{aligned}$$

the last form following from

$$\frac{\partial}{\partial x}|\mathbf{r} - \mathbf{r}'| = -\frac{\partial}{\partial x'}|\mathbf{r} - \mathbf{r}'|$$

Now

$$\int d\mathbf{r}' \left\{\frac{\partial}{\partial x'}\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right\} j_x(\mathbf{r}', t - \tau) = -\int \frac{d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}\frac{\partial}{\partial x'} j_x(\mathbf{r}', t - \tau)$$

Therefore

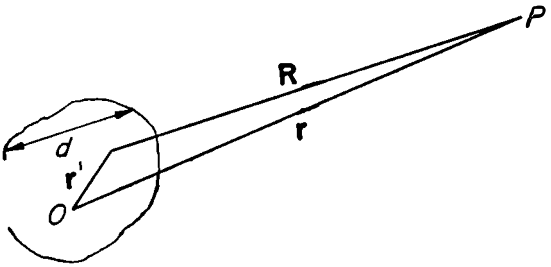
$$\begin{aligned}
\frac{\partial \psi_x}{\partial x} &= \int \frac{d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}\left\{\frac{\partial}{\partial x'}\{j_x(\mathbf{r}', t - \tau)\} - \frac{\partial}{\partial \tau} j_x(\mathbf{r}', t - \tau)\frac{\partial \tau}{\partial x'}\right\} \\
&= \int \frac{d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}\left[\frac{\partial}{\partial x'} j_x(\mathbf{r}', t')\right]_{t' = t - \tau}
\end{aligned}$$

and hence

$$\frac{1}{c}\frac{\partial\varphi}{\partial t} + \nabla\cdot\boldsymbol{\psi}$$
$$= \int \frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\left[\frac{1}{c}\frac{\partial}{\partial t}\varrho(\mathbf{r}', t') + \nabla'\cdot\mathbf{j}(\mathbf{r}', t')\right]_{t'=t-\tau}$$
$$= 0$$

q.e.d.

Let us now consider some particular cases of our solutions. Suppose the charge and current distribution are confined to a region of dimension d, and consider the radiation at a point P outside it.



For $\dfrac{R}{c} \equiv \dfrac{|\mathbf{r}-\mathbf{r}'|}{c}$ small, the solutions reduce to

$$\varphi(\mathbf{r}, t) \approx \int \frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\varrho(\mathbf{r}', t)$$
$$\boldsymbol{\psi}(\mathbf{r}, t) \approx \int \frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\mathbf{j}(\mathbf{r}', t) \tag{7.9}$$

These solutions are good if, during a length of time R/c, ϱ and $\mathbf{j}$ do not change appreciably. (That is,

$$\frac{R}{c} \ll \frac{1}{\omega} = T$$

where ω is a typical frequency in the Fourier decompositions of ϱ and $\mathbf{j}$.) They are called the quasi-static solutions and their fields are the ones given by Coulomb's and Ampere's Laws.

For large values of R, the approximate solutions give the wavezone field. The condition of validity cannot be given in a general way but only for the Fourier components. It is

$$\frac{\nu R}{c} \gg 1$$

We shall first show that for $\dfrac{\nu R}{c} \gg 1$ and $r \gg r'$

$$\varphi_\nu \approx \mathbf{n} \cdot \boldsymbol{\psi}_\nu$$

where, under our second condition,

$$\mathbf{n} \approx \frac{\mathbf{r}}{r}$$

From the Lorentz condition we have

$$\nabla \cdot \boldsymbol{\psi}_\nu + \frac{i\nu}{c}\varphi_\nu = 0$$

Now

$$\frac{\partial \psi_{\nu x}}{\partial x} = \frac{\partial}{\partial x}\int \frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} j_{\nu x}(\mathbf{r}')\, e^{-i\frac{\nu}{c}|\mathbf{r}-\mathbf{r}'|}$$

and

$$|\mathbf{r}-\mathbf{r}'| = r - r'\cos(\mathbf{r}, \mathbf{r}') + 0\left(\frac{r'^2}{r}\right) \approx r - \mathbf{n}\cdot\mathbf{r}'$$

Therefore

$$\frac{\partial \psi_{\nu x}}{\partial x} \approx \frac{1}{r}\int d\mathbf{r}'\, j_{\nu x}(\mathbf{r}')\left(-i\frac{\nu}{c}\right)\frac{x}{r}e^{-i\frac{\nu}{c}|\mathbf{r}-\mathbf{r}'|}$$

where we have kept only the term with the lowest power of $1/r$. By a similar approximation it follows that

$$\frac{\partial \psi_{\nu x}}{\partial x} \approx -\frac{i\nu}{c} n_x \psi_{\nu x}$$

and hence

$$\nabla \cdot \boldsymbol{\psi}_\nu \approx -i\frac{\nu}{c}\mathbf{n}\cdot\boldsymbol{\psi}_\nu$$

The Lorentz condition then yields

$$\varphi_\nu \approx \mathbf{n} \cdot \boldsymbol{\psi}_\nu$$

In the following considerations, we shall restrict our attention to one frequency ν so that

$$\mathbf{j}(\mathbf{r}, t) = \mathbf{j}_\nu(\mathbf{r})\, e^{i\nu t} + \bar{\mathbf{j}}_\nu(\mathbf{r})\, e^{-i\nu t}$$

Now

$$\varphi_\nu(\mathbf{r}) \approx \frac{e^{-i\frac{\nu}{c}r}}{r} \int d\mathbf{r}'\, \varrho_\nu(\mathbf{r}')\, e^{i\frac{\nu}{c}\mathbf{n}\cdot\mathbf{r}'}$$

$$\boldsymbol{\psi}_\nu(\mathbf{r}) \approx \frac{e^{-i\frac{\nu}{c}r}}{r} \int d\mathbf{r}'\, \mathbf{j}_\nu(\mathbf{r}')\, e^{i\frac{\nu}{c}\mathbf{n}\cdot\mathbf{r}'}$$

and

$$\mathbf{E}_\nu = -\nabla\varphi_\nu - \frac{i\nu}{c}\boldsymbol{\psi}_\nu$$

$$\mathbf{H}_\nu = \nabla \times \boldsymbol{\psi}_\nu$$

First we shall show that $\mathbf{E} \perp \mathbf{H} \perp \mathbf{n}$. In carrying out the differentiation of the potentials, we shall keep only the terms which are proportional to $1/r$ and neglect terms in higher powers of $1/r$, since we are interested in the value of the field far away from the charges. Let

$$\gamma = \int d\mathbf{r}'\, \mathbf{j}_\nu(\mathbf{r}')\, e^{i\frac{\nu}{c}\mathbf{n}\cdot\mathbf{r}'}$$

Then

$$(\nabla \times \boldsymbol{\psi}_\nu)_z \approx \left\{\left(-\frac{i\nu}{c}n_x\right)\gamma_y - \left(-i\frac{\nu}{c}n_y\right)\gamma_x\right\}\frac{e^{-i\frac{\nu}{c}r}}{r}$$

$$\mathbf{E}_\nu \approx \frac{i\nu}{c}\mathbf{n}\varphi_\nu - \frac{i\nu}{c}\boldsymbol{\psi}_\nu$$

$$\mathbf{H}_\nu \approx \frac{i\nu}{rc}\gamma \times \mathbf{n}\, e^{\frac{-i\nu r}{c}}$$

$$\mathbf{E}_\nu \approx -\frac{i\nu}{c}\{\boldsymbol{\psi}_\nu - \mathbf{n}(\mathbf{n}\cdot\boldsymbol{\psi}_\nu)\}$$

$$\approx -\frac{i\nu}{cr}e^{-i\frac{\nu r}{c}}\{\gamma - \mathbf{n}(\mathbf{n}\cdot\gamma)\}$$

Now

$$\mathbf{n} \times \mathbf{H}_\nu \approx \frac{i\nu}{rc}\mathbf{n} \times [\gamma \times \mathbf{n}]\, e^{-\frac{i\nu}{c}r}$$

$$\approx \frac{i\nu}{rc}\{\gamma - \mathbf{n}(\mathbf{n}\cdot\gamma)\}\, e^{-i\frac{\nu}{c}r}$$

$$\approx -\mathbf{E}_\nu$$

Hence $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{n}$ are perpendicular to each other and form a righthanded system in this order. Also $|\mathbf{E}| = |\mathbf{H}|$.

Next we shall show that the potentials

$$\boldsymbol{\psi}'_\nu = \boldsymbol{\psi}_\nu - \mathbf{n}(\mathbf{n} \cdot \boldsymbol{\psi}_\nu)$$
$$\varphi'_\nu = 0$$

give the same field as $\boldsymbol{\psi}_\nu$ and φ_ν in the present approximation:

$$\begin{aligned} \mathbf{E}'_\nu &= -\frac{i\nu}{c}\{\boldsymbol{\psi}_\nu - \mathbf{n}(\mathbf{n} \cdot \boldsymbol{\psi}_\nu)\} \\ &\approx -\frac{i\nu}{c}\{\boldsymbol{\psi}_\nu - \mathbf{n}\varphi_\nu\} \\ &\approx \mathbf{E}_\nu \\ \mathbf{H}'_\nu &= \nabla \times \{\boldsymbol{\psi}_\nu - \mathbf{n}(\mathbf{n} \cdot \boldsymbol{\psi}_\nu)\} \\ &\approx \nabla \times \boldsymbol{\psi}_\nu + \mathbf{n} \times \nabla \varphi_\nu \\ &\approx \mathbf{H}_\nu \end{aligned}$$

EXERCISE 12 Taking ϱ and $\mathbf{j}$ for the hydrogen atom given in Exercise 11, calculate to what distance the quasi-static field and from what distance the wave zone field are good to within 10 percent of the correct field.

Let us now consider the Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}$$

From the above considerations we have

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{n}\,|\mathbf{E}|^2$$

where

$$\mathbf{E} = \mathbf{E}_\nu e^{i\nu t} + \mathbf{E}_\nu e^{-i\nu t}$$
$$|\mathbf{E}|^2 = \mathbf{E}_\nu^2 e^{2i\nu t} + \mathbf{E}_\nu^2 e^{-2i\nu t} + 2|\mathbf{E}_\nu|^2$$

On the average, the oscillating terms are zero, so that

$$\begin{aligned} \mathbf{S} &= \frac{c}{2\pi}\mathbf{n}\,\frac{\nu^2}{c^2 r^2}\,|\boldsymbol{\gamma} - \mathbf{n}(\mathbf{n} \cdot \boldsymbol{\gamma})|^2 \\ &= \frac{\nu^2 \mathbf{n}}{2\pi c r^2}\left|\int d\mathbf{r}'\, \{\mathbf{j}(\mathbf{r}') - \mathbf{n}(\mathbf{n} \cdot \mathbf{j}(\mathbf{r}'))\}\, e^{-\frac{i\nu}{c}(\mathbf{n} \cdot \mathbf{r})}\right|^2 \end{aligned}$$

Hence the rate of radiation in the solid angle $d\boldsymbol{\Omega}$ is given by

$$dR = d\boldsymbol{\Omega}\frac{\nu^2}{2\pi c}\left|\int d\mathbf{r}'\,\mathbf{j}_\perp(\mathbf{r}')\,e^{-\frac{i\nu}{c}(\mathbf{n}\cdot\mathbf{r}')}\right|^2 \tag{7.10}$$

which is the same as (6.12) calculated in Section 6.

We shall finally consider the general case, and calculate the energy radiated by a pulse of current. We have

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}$$

$$= \frac{c}{4\pi}\int_{-\infty}^{+\infty} d\nu \int_{-\infty}^{+\infty} d\nu'\,\mathbf{E}\times\mathbf{H}_{\nu'}e^{i(\nu+\nu')t}$$

$$\int_{-\infty}^{\infty}\mathbf{S}\,dt = \frac{c}{2}\int_{-\infty}^{+\infty} d\nu \int_{-\infty}^{+\infty} d\nu'\,\mathbf{E}_\nu\times\mathbf{H}_{\nu'}\,\delta(\nu+\nu')$$

$$= \frac{c}{2}\int_{-\infty}^{+\infty} d\nu\,\mathbf{E}_\nu\times\mathbf{H}_{-\nu}$$

We have seen that in the wave zone field

$$\mathbf{H}_\nu = \frac{i\nu}{rc}\gamma_\nu\times\mathbf{n}\,e^{-\frac{i\nu}{c}r}$$

where

$$\gamma_\nu = \int \mathbf{j}_\nu(\mathbf{r}')e^{\frac{i\nu}{c}(\mathbf{n}\cdot\mathbf{r}')}\,d\mathbf{r}'$$

and

$$\mathbf{E}_\nu = \mathbf{H}_\nu\times\mathbf{n},\quad \mathbf{E}_\nu\times\mathbf{H}_{-\nu} = \mathbf{n}\mathbf{H}_\nu\cdot\mathbf{H}_{-\nu}$$

Hence, restricting ν to positive values

$$\int_{-\infty}^{\infty}\mathbf{S}\,dt = c\frac{1}{c^2r^2}\mathbf{n}\int_0^{\infty} d\nu\,\nu^2\,|\gamma_\nu\times\mathbf{n}|^2$$

and if Q_ν is the total radiation emitted per unit frequency range, then

$$\int_0^\infty Q_\nu d\nu = \int d\Omega \int_{-\infty}^{\infty} dt\, r^2 \mathbf{S}\cdot\mathbf{n}$$

$$= \frac{1}{c}\int d\Omega \int_0^\infty d\nu\, \nu^2\, |\boldsymbol{\gamma}_\nu \times \mathbf{n}|^2$$

$$dQ_\nu\, d\nu = \frac{\nu^2}{c}\, d\Omega\, d\nu\, |\boldsymbol{\gamma}_\nu \times \mathbf{n}|^2$$

Putting in the value for $\boldsymbol{\gamma}_\nu$ we have

$$dQ_\nu\, d\nu = \frac{\nu^2}{c}\, d\Omega\, d\nu \left| \int d\mathbf{r}\, [\mathbf{n} \times \mathbf{j}_\nu(\mathbf{r})]\, e^{\frac{i\nu}{c}(\mathbf{n}\cdot\mathbf{r})} \right|^2$$

$$= \frac{\nu^2}{4\pi^2 c}\, d\Omega\, d\nu \left| \int d\mathbf{r} \int dt\, [\mathbf{n} \times \mathbf{j}(\mathbf{r}, t)]\, e^{-i(\nu t - \frac{\nu}{c}\mathbf{n}\cdot\mathbf{r})} \right|^2 \tag{7.11}$$

8 CLASSIFICATION OF MULTIPOLE RADIATION

We have seen that

$$\mathbf{E}_\nu \approx -\frac{i\nu}{c}\{\boldsymbol{\psi}_\nu - \mathbf{n}(\mathbf{n}\cdot\boldsymbol{\psi}_\nu)\}$$

$$\mathbf{H}_\nu \approx -\frac{i\nu}{c}\mathbf{n} \times \boldsymbol{\psi}_\nu$$

where

$$\boldsymbol{\psi}_\nu = \frac{e^{-\frac{\nu}{c}r}}{r}\int d\mathbf{r}'\, \mathbf{j}_\nu(\mathbf{r}')\, e^{i\frac{\nu}{c}\mathbf{n}\cdot\mathbf{r}'}$$

Now if d is the size of the charge distribution and

$$\frac{\nu d}{c} = \frac{d}{\lambda} < 1$$

we can expand the exponential. This condition can also be stated in the form

$$\frac{v_{\text{charge}}}{c} < 1$$

since $\nu d \sim v_{\text{charge}}$. Also for atoms, we have from the uncertainty principle

$$d \sim \frac{\hbar}{\sqrt{\mathbf{p}^2}}$$

$d\sqrt{\mathbf{p}^2}$ can be greater than $\hbar$ but it will be of this order for the lower states. Hence

$$\frac{p_{\text{rad}}}{p_{\text{charge}}} \sim \frac{\hbar \dfrac{\nu}{c}}{\hbar/d} = \frac{\nu d}{c}$$

Hence another form of expressing the condition is

$$\frac{p_{\text{rad}}}{p_{\text{charge}}} < 1$$

Thus if any one of the three equivalent conditions obtain

1) $d \ll \lambda$

2) $v_{\text{charge}} \ll c$

3) $p_{\text{rad}} \ll p_{\text{charge}}$

then it will be sufficient to expand the exponential and take the first few terms. The condition holds for optical transitions in atoms, γ-ray emission from nuclei, and for some macroscopic systems. Taking just the first term of the exponential, that is, replacing it by unity, corresponds to leaving out the recoil of the charge since then the absolute values of **E** and **H** are not changed by replacing **n** by $-\mathbf{n}$. It also corresponds to leaving out the magnetic field of the charge in motion since taking the exponent zero means taking $v_{\text{charge}} = 0$. And finally it corresponds to neglecting the phase difference between radiations emitted from different parts of the charge distribution.

The usual procedure in making the calculation is to expand the exponential

$$e^{i\frac{\nu}{c}\mathbf{n}\cdot\mathbf{r}'} = \sum_k \left\{\frac{i\nu}{c}(\mathbf{n}\cdot\mathbf{r}')\right\}^k \frac{1}{k!}$$

and take the first non-vanishing term. An exception occurs in the nucleus where there are approximately equal numbers of protons and neutrons. It cannot oscillate as a whole and it cannot vary its charge distribution very much, so its dipolo moment is small. However, it can have a large quadrupole moment by deforming its shape, and it turns out that the quadrupole radiation is of the same order of magnitude as the dipole radiation, although vd/c is quite small.

The classification of radiation is done in two ways:

1) by k, the power of the exponent,
2) by the angular distribution of the radiation.

a) Electric dipole radiation ($k = 0$)

For this case

$$e_l = \int d\mathbf{r}' \left\{ j_{\nu l}(\mathbf{r}') - n_l \sum_j j_{\nu j}(\mathbf{r}')\, n_j \right\}$$

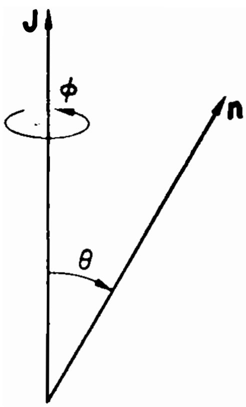
$$h_l = \int d\mathbf{r}' \left\{ n_j j_{\nu i}(\mathbf{r}') - n_i j_{\nu j}(\mathbf{r}') \right\} \tag{8.1}$$

where i, j, l form a cyclic permutation of x, y, z, and where

$$\mathbf{e} = -\frac{c}{i\nu} r e^{i\frac{\nu r}{c}} \mathbf{E}_\nu \qquad \mathbf{h} = -\frac{c}{i\nu} r e^{i\frac{\nu r}{c}} \mathbf{H}_\nu \tag{8.2}$$

Let us take the z-axis parallel to*

$$\mathbf{J} \equiv \int d\mathbf{r}'\, \mathbf{j}_\nu(\mathbf{r}')$$



* The relation of the vector $\mathbf{J}$ to the electric dipole moment may be made apparent by using the charge conservation law $\nabla \cdot \mathbf{j}_\nu + \dfrac{i\nu}{c} \varrho_\nu = 0$. From Gauss' theorem we have

$$0 = \int \nabla \cdot (\mathbf{j}_\nu \mathbf{r})\, d\mathbf{r} = -\frac{i\nu}{c} \int \mathbf{r}\, \varrho_\nu\, d\mathbf{r} + \int \mathbf{j}_\nu\, d\mathbf{r}$$

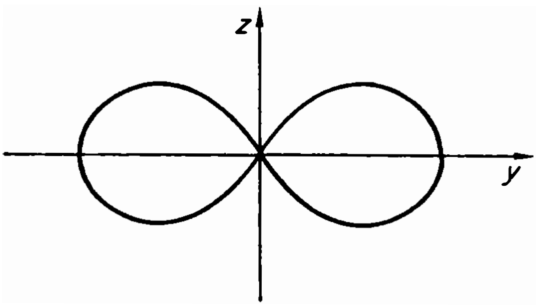
whence $\qquad \mathbf{J} = \dfrac{i\nu}{c} \mathbf{P} \qquad$ with $\qquad \mathbf{P} \equiv \displaystyle\int \mathbf{r} \varrho_\nu\, d\mathbf{r}.$ [Ed.]

Then

$$
\begin{aligned}
n_x &= \sin\theta\cos\varphi \\
n_y &= \sin\theta\sin\varphi \\
n_z &= \cos\theta \\
J_x &= 0 \\
J_y &= 0 \\
J_z &= J \\
e_x &= -J\sin\theta\cos\theta\cos\varphi \\
e_y &= -J\sin\theta\cos\theta\sin\varphi \\
e_z &= J\sin^2\theta \\
h_x &= J\sin\theta\sin\varphi \\
h_y &= -J\sin\theta\cos\varphi \\
h_z &= 0
\end{aligned}
$$

and hence the angular distribution of the radiation is

$$S \propto \sin^2\theta$$



b) Magnetic dipole and electric quadrupole radiations ($k = 1$)

In this case

$$e_l = \frac{i\nu}{c}\int d\mathbf{r}' \sum_s n_s r'_s \left\{ j_{\nu l}(\mathbf{r}') - n_i \sum_j j_{\nu j}(\mathbf{r}')\, n_j \right\} \tag{8.3}$$

$$h_i = \frac{i\nu}{c}\int d\mathbf{r}' \sum_s n_s r'_s \{ n_j j_{\nu l}(\mathbf{r}') - n_l j_{\nu j}(\mathbf{r}') \}$$

Let

$$\gamma_{is} = \frac{iv}{c}\int d\mathbf{r}'\, j_{\nu i}(\mathbf{r}')\, r'_s$$

Then

$$\left.\begin{aligned} e_i &= \sum_s \gamma_{is} n_s - \sum_{j,s} \gamma_{js} n_s n_i n_j \\ h_i &= \sum_s \gamma_{ls} n_s n_j - \sum_s \gamma_{js} n_s n_l \qquad (i, j, l) \end{aligned}\right\} \tag{8.4}$$

Now γ_{is} is a tensor, and it can be broken up into symmetric and antisymmetric parts:

$$\gamma_{is} = \tfrac{1}{2}(\gamma_{is} + \gamma_{si}) + \tfrac{1}{2}(\gamma_{is} - \gamma_{si})$$

The symmetric part gives the electric quadrupole radiation, and the antisymmetric part the magnetic dipole radiation.

First let us consider the magnetic dipole radiation. Let

$$m_t = \frac{1}{2}(\gamma_{is} - \gamma_{si}) = \frac{iv}{c}\int [\mathbf{j}_\nu \times \mathbf{r}']_t\, d\mathbf{r}'$$

where t, i, s form a cyclic permutation of x, y, z. We see that $\mathbf{m}$ is the moment of the current. Replacing γ_{is} by m_t in (8.4) and using the antisymmetric properties, we get

$$e_i = [\mathbf{n} \times \mathbf{m}]_i$$

$$h_i = n_i(\mathbf{n} \cdot \mathbf{m}) - m_i$$

Comparing these equations with (8.1), we see that the field of the magnetic dipole may be obtained from that of the electric dipole by making the correspondences

$$\mathbf{h}_e \to \mathbf{e}_m, \qquad \mathbf{e}_e \to -\mathbf{h}_m, \qquad \mathbf{J} \to \mathbf{m}$$

Hence the angular distribution of the radiation is again given by

$$S \propto \sin^2\theta$$

However the magnitude of S is in general smaller by the factor $\left(\dfrac{d}{\lambda}\right)^2$.

Now

$$\int [\mathbf{j} \times \mathbf{r}']\, d\mathbf{r}' = \frac{1}{c}\int \varrho[\mathbf{v} \times \mathbf{r}']\, d\mathbf{r}'$$

and if the mass density μ is proportional to the charge density ϱ, the magnetic dipole moment is proportional to the moment of mementum

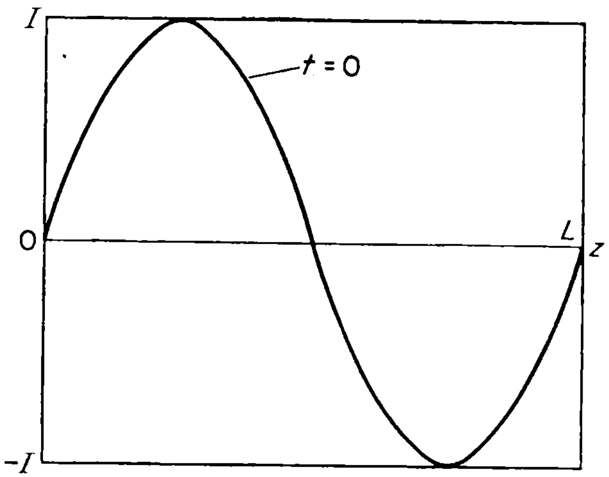
$$\int \mu[\mathbf{v} \times \mathbf{r}']\, d\mathbf{r}'$$

Since this is constant for an isolated system, it cannot have magnetic dipole radiation. This would be true for electrons in atoms if the electron did not have a spin with a g-factor of approximately 2 rather than unity. A ring of current which varies harmonically with time has magnetic dipole moment. However, such oscillation cannot occur in an isolated system if μ/ϱ is constant, for the angular momentum of such a system cannot oscillate.

Let us now consider the electric quadrupole moment.* The symmetric part of γ_{is} has 6 components, but only 5 independent quantities determine this radiation. This follows from the fact that the values of **e** and **h** are unaltered by an addition of diagonal terms $\gamma\,\delta_{is}$ to γ_{is}, and this is related to the fact that the radial component of the current does not give rise to any radiation.

Consider the current distribution

$$\mathbf{j}_\nu = \frac{\mathbf{z}}{z}\,\frac{I}{2}\sin\frac{2\pi z}{L}\,\delta(x)\,\delta(y)$$



For this case

$$\gamma_{zz} = Q = \frac{i\nu}{c}\int \mathbf{j}_{\nu z} z\, d\mathbf{r}$$

* Again from Gauss' theorem we have $0 = \int \nabla\cdot(\mathbf{j}_\nu \mathbf{r}\mathbf{r})\,d\mathbf{r} = -\frac{i\nu}{c}\int \varrho_\nu \mathbf{r}\mathbf{r}\,d\mathbf{r} + \int (\mathbf{j}_\nu\mathbf{r} + \mathbf{r}\mathbf{j}_\nu)\,d\mathbf{r}$ whence it follows that $\gamma_{ij}^{\text{sym}} - \frac{1}{3}\delta_{ij}\sum_k \gamma_{kk} = \frac{1}{2}\left(\frac{i\nu}{c}\right)^2 Q_{ij}$ where Q_{ij} is the quadrupole moment tensor:

$$Q_{ij} = \int \varrho_\nu\left(r_i r_j - \frac{1}{3}\delta_{ij} r^2\right) d\mathbf{r}. \qquad \text{[Ed.]}$$

and all other $\gamma_{is} = 0$. If the polar axis is taken along the z-axis as usual, we have

$$e_x = -Q \sin\theta \cos^2\theta \cos\varphi$$

$$e_y = -Q \sin\theta \cos^2\theta \sin\varphi$$

$$e_z = Q \sin^2\theta \cos$$

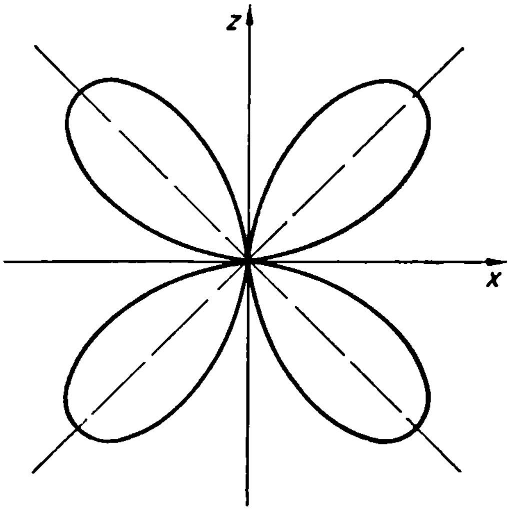
$$h_x = Q \sin\theta \cos\theta \sin\varphi$$

$$h_y = -Q \sin\theta \cos\theta \cos\varphi$$

$$h_z = 0$$

and the angular distribution is

$$S \propto \sin^2 2\theta$$



EXERCISE 13 Calculate the angular distribution of the radiation from the following current system

$$\mathbf{j}_v = \frac{\mathbf{z}}{z}\frac{I}{2}\left\{\sin\frac{\pi z}{L}\,\delta(x)\delta(y) - \sin\frac{\pi z}{L}\,\delta(x+l)\,\delta(y)\right\}$$

This represents a harmonically oscillating current in two linear conductors of length L parallel to the z axis and a small distance l apart. It will be found that only γ_{zx} is different from zero.

EXERCISE 14 In Exercise 10, the charge accumulated on one-half of the wire for the case of dipole radiation was calculated. Make a similar calculation for the quadrupole radiation using the same dimensions and numbers, and compare the two results.

If we can find the angular distribution of radiation from atoms, then we can tell what kind of radiation it is and hence learn something about the atom. However, we cannot find it by measuring the intensity of the radiation at different directions since we have to have many atoms and these would be oriented at random. The radiation from a single atom is coherent, but radiations from different atoms are incoherent. It has been suggested that the study of the interference properties of the radiation from atoms would give the type of radiation.

The character of γ-radiation from nuclei may be found by studying the field at a distance r from the nucleus such that

$$r \gg d$$

$$r < \lambda$$

This region is called the diffraction zone. Under these conditions, we can still expand the exponential

$$e^{i\frac{\nu}{c}|\mathbf{r}-\mathbf{r}'|}$$

and we can also expand

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\sum_k \left(\frac{r'}{r}\right)^k P_k(\cos(\mathbf{r}, \mathbf{r}'))$$

Thus the electric dipole field falls off as $1/r$, the magnetic dipole and electric quadrupole field as $1/r^2$, etc. Hence the fields of the higher multipoles become relatively larger as we go nearer the nucleus. Now there are electrons around the nucleus, and their reactions to the field tell us the character of the radiation. In the actual calculation it is found more convenient to use the expansion

$$e^{\frac{i\nu}{c}(\mathbf{n}\cdot\mathbf{r}')} = \sum_l P_l(\cos(\mathbf{n}, \mathbf{r}'))f_l(r')$$

where

$$f_l(r) = (2l+1)\frac{i^l}{\sqrt{\frac{\nu r}{c}}}\sqrt{\frac{\pi}{2}}J_{l+1/2}\left(\frac{\nu r}{c}\right) \approx \left(\frac{\nu r}{c}\right)^l$$

instead of the power-series expansion.

The fields of higher multipoles become increasingly more complicated, and we shall not go into their calculation.

9 ENERGY OF A NEARLY STATIC DISTRIBUTION OF CHARGE

In this calculation we take $\mathbf{j}$ to be static. If we took ϱ to be static also, then we would just get the electrostatic formula. Hence we ease the condition and say that ϱ varies with t but only in such a way that $\mathbf{j}$ is static. We want to calculate

$$E = \frac{1}{8\pi}\int d\mathbf{r}(\mathbf{E}^2 + \mathbf{H}^2)$$

under this condition. We take the potentials φ, $\mathbf{A}$ such that

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{H} = \nabla\times\mathbf{A}$$

$$\nabla\cdot\mathbf{A} = 0$$

Then as in Section 6 we have

$$\mathbf{E}_{\parallel} = -\nabla\int\frac{\varrho(\mathbf{r}')\,d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \Delta\mathbf{A} = 4\pi\mathbf{j}_{\perp}$$

$$\mathbf{j}_{\perp} = \mathbf{j} + \frac{1}{4\pi}\nabla\int\frac{\nabla'\cdot\mathbf{j}(\mathbf{r}')\,d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

Now since $\mathbf{j}$ is static, $\mathbf{j}_{\perp}$ is also, and hence $\mathbf{A}$ must be static. That is, $\dfrac{\partial\mathbf{A}}{\partial t} = 0$, and therefore

$$\mathbf{A} = \int d\mathbf{r}'\,\frac{\mathbf{j}_{\perp}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$E_{\mathrm{E}} = \frac{1}{8\pi}\int\mathbf{E}^2\,d\mathbf{r}$$

$$= \frac{1}{8\pi}\int d\mathbf{r}\,\nabla\varphi\cdot\nabla\varphi$$

By partial integration we obtain

$$E_{\mathrm{E}} = -\frac{1}{8\pi}\int d\mathbf{r}\,\varphi\,\Delta\varphi$$

$$= -\frac{1}{8\pi}\int d\mathbf{r}\left[\left\{\int d\mathbf{r}'\,\frac{\varrho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right\}\left\{\Delta\int d\mathbf{r}''\,\frac{\varrho(\mathbf{r}'')}{|\mathbf{r}-\mathbf{r}''|}\right\}\right.$$

and since

$$\Delta \frac{1}{|\mathbf{r} - \mathbf{r}''|} = -4\pi\,\delta(\mathbf{r} - \mathbf{r}'')$$

we have finally

$$E_{\mathrm{E}} = \frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'\,\frac{\varrho(\mathbf{r}')\,\varrho(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|} \tag{9.1}$$

Next

$$E_{\mathrm{H}} = \frac{1}{8\pi}\int \mathbf{H}^2\,d\mathbf{r}$$

$$= \frac{1}{8\pi}\int (\nabla \times \mathbf{A})\cdot(\nabla \times \mathbf{A})\,d\mathbf{r}$$

$$= \frac{1}{8\pi}\int d\mathbf{r}\left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)^2 + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)^2 + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)^2\right]$$

If we expand the integrand and then integrate partially, we get

$$\begin{aligned} E_{\mathrm{H}} = -\frac{1}{8\pi}\int d\mathbf{r}\Big[&A_z\frac{\partial^2 A_z}{\partial y^2} - A_y\frac{\partial^2 A_z}{\partial z\,\partial y} - A_z\frac{\partial^2 A_y}{\partial z\,\partial y} + A_y\frac{\partial^2 A_y}{\partial z^2} \\ &+ A_x\frac{\partial^2 A_x}{\partial z^2} - A_z\frac{\partial^2 A_x}{\partial x\,\partial z} - A_x\frac{\partial^2 A_z}{\partial x\,\partial z} + A_z\frac{\partial^2 A_z}{\partial x^2} \\ &+ A_y\frac{\partial^2 A_y}{\partial x^2} - A_x\frac{\partial^2 A_y}{\partial y\,\partial x} - A_y\frac{\partial^2 A_x}{\partial y\,\partial x} + A_x\frac{\partial^2 A_x}{\partial y^2}\Big] \end{aligned}$$

$$= -\frac{1}{8\pi}\int d\mathbf{r}\,[\mathbf{A}\cdot\Delta\mathbf{A} - \mathbf{A}\cdot\nabla(\nabla\cdot\mathbf{A})]$$

Now $\nabla\cdot\mathbf{A} = 0$, so

$$E_{\mathrm{H}} = -\frac{1}{8\pi}\int d\mathbf{r}\left[\left\{\int\frac{d\mathbf{r}'\,\mathbf{j}_\perp(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\right\}\cdot\left\{\Delta\int\frac{d\mathbf{r}''\,\mathbf{j}_\perp(\mathbf{r}'')}{|\mathbf{r} - \mathbf{r}''|}\right\}\right]$$

$$= \frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'\,\frac{\mathbf{j}_\perp(\mathbf{r}')\cdot\mathbf{j}_\perp(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|} \tag{9.2}$$

If $\dfrac{\partial\varrho}{\partial t} = 0$ so that $\nabla\cdot\mathbf{j} = 0$, then $\mathbf{j}_\perp = \mathbf{j}$, and we obtain the familiar expression for the energy of a magnetic field due to steady currents

$$E_{\mathrm{H}} = \frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'\,\frac{\mathbf{j}(\mathbf{r}')\cdot\mathbf{j}(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|}$$

However, if $\dfrac{\partial \varrho}{\partial t} \neq 0$, then we get a more complicated expression. We have

$$\mathbf{j}_{\perp} = \mathbf{j} + \frac{1}{4\pi} \nabla \chi$$

where

$$\chi = \int d\mathbf{r}'' \frac{\nabla'' \cdot \mathbf{j}(\mathbf{r}'')}{|\mathbf{r} - \mathbf{r}''|}$$

which by partial integration gives

$$\chi = -\int d\mathbf{r}'' \, \mathbf{j}(\mathbf{r}'') \cdot \nabla'' \frac{1}{|\mathbf{r} - \mathbf{r}''|}$$

$$\nabla \chi = \int d\mathbf{r}'' \, (\mathbf{j}(\mathbf{r}'') \cdot \nabla'') \, \nabla'' \frac{1}{|\mathbf{r} - \mathbf{r}''|}$$

From (9.2) we get

$$E_{\mathbf{H}} = \frac{1}{2} \int d\mathbf{r} \int d\mathbf{r}' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \left\{ \mathbf{j}(\mathbf{r}) + \frac{1}{4\pi} \nabla \chi \right\} \cdot \left\{ \mathbf{j}(\mathbf{r}') + \frac{1}{4\pi} \nabla' \chi(\mathbf{r}') \right\}$$

$$= \frac{1}{2} I_1 + \frac{1}{4\pi} I_2 + \frac{1}{32\pi^2} I_3$$

where

$$I_1 = \int d\mathbf{r} \int d\mathbf{r}' \frac{\mathbf{j}(\mathbf{r}) \cdot \mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$I_2 = \int d\mathbf{r} \int d\mathbf{r}' \frac{\mathbf{j}(\mathbf{r}') \cdot \nabla \chi(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|}$$

$$I_3 = \int d\mathbf{r} \int d\mathbf{r}' \frac{\nabla \chi(\mathbf{r}) \cdot \nabla' \chi(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Now

$$I_2 = \int d\mathbf{r} \int d\mathbf{r}' \int d\mathbf{r}'' \frac{(\mathbf{j}(\mathbf{r}'') \cdot \nabla'')}{|\mathbf{r} - \mathbf{r}'|} \mathbf{j}(\mathbf{r}') \cdot \nabla'' \frac{1}{|\mathbf{r} - \mathbf{r}''|}$$

$$= \int d\mathbf{r}' \int d\mathbf{r}'' \, \mathbf{j}(\mathbf{r}') \cdot \mathbf{F}(\mathbf{r}' - \mathbf{r}'') \cdot \mathbf{j}(\mathbf{r}'')$$

where

$$\mathbf{F}(\mathbf{r}' - \mathbf{r}'') \equiv \int \frac{d\mathbf{r}}{|\mathbf{r} - \mathbf{r}'|} \nabla'' \nabla'' \frac{1}{|\mathbf{r} - \mathbf{r}''|}$$

This tensor function is the unique solution of the differential equation

$$\Delta \mathbf{F}(\mathbf{r}) = -4\pi \, \nabla \nabla \frac{1}{r}$$

which vanishes as $r \to \infty$. Introducing the unit tensor

$$\mathbf{1} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we may reexpress the right hand side of this equation in the form

$$\Delta \mathbf{F}\,(\mathbf{r}) = -4\pi \left[3\frac{\mathbf{rr}}{r^5} - \frac{\mathbf{1}}{r^3} - \frac{4\pi}{3}\, \mathbf{1}\delta(\mathbf{r}) \right]$$

the delta function term being included so that we obtain

$$\Delta \operatorname{tr} \mathbf{F} = -4\pi \Delta \frac{1}{r} = (-4\pi)^2 \, \delta(\mathbf{r})$$

We try a solution of the form

$$\mathbf{F}(\mathbf{r}) = a\frac{\mathbf{1}}{r} + b\frac{\mathbf{rr}}{r^3}$$

By straightforward computation we find

$$\Delta \, \mathbf{F}(\mathbf{r}) = -4\pi a \mathbf{1}\delta(\mathbf{r}) + 2b\frac{\mathbf{1}}{r^3} - 6b\frac{\mathbf{rr}}{r^5} - \frac{4\pi}{3}\, b\mathbf{1}\delta(\mathbf{r})$$

whence it follows that

$$a = -b = -2\pi$$

and hence

$$I_2 = -2\pi \int d\mathbf{r}' \int d\mathbf{r}'' \left\{ \frac{\mathbf{j}(\mathbf{r}') \cdot \mathbf{j}(\mathbf{r}'')}{|\mathbf{r}' - \mathbf{r}''|} - \frac{\mathbf{j}(\mathbf{r}') \cdot (\mathbf{r}' - \mathbf{r}'')\, \mathbf{j}(\mathbf{r}'') \cdot (\mathbf{r}' - \mathbf{r}'')}{|\mathbf{r}' - \mathbf{r}''|^3} \right\}$$

I_3 may be written in the form

$$I_3 = \int d\mathbf{r} \int d\mathbf{r}' \int d\mathbf{r}'' \int d\mathbf{r}''' \frac{\mathbf{j}(\mathbf{r}'') \cdot \nabla'' \, \mathbf{j}(\mathbf{r}''' \cdot \nabla''')}{|\mathbf{r} - \mathbf{r}'|} \nabla'' \frac{1}{|\mathbf{r} - \mathbf{r}''|} \cdot \nabla''' \frac{1}{|\mathbf{r}' - \mathbf{r}'''|}$$

$$= \int d\mathbf{r}'' \int d\mathbf{r}''' \, \mathbf{j}(\mathbf{r}'') \cdot \mathbf{G}(\mathbf{r}'' - \mathbf{r}''') \cdot \mathbf{j}(\mathbf{r}''')$$

where

$$\mathbf{G}(\mathbf{r}'' - \mathbf{r}''') \equiv \int d\mathbf{r} \int d\mathbf{r}' \, \nabla'' \, \nabla''' \left[\frac{1}{|\mathbf{r} - \mathbf{r}'|} \left(\nabla'' \frac{1}{|\mathbf{r} - \mathbf{r}''|} \cdot \nabla''' \frac{1}{|\mathbf{r}' - \mathbf{r}'''|} \right) \right]$$

Now, applying the Laplacian operator and integrating by parts, we get

$$\begin{aligned}
\Delta''\mathbf{G}(\mathbf{r}''-\mathbf{r}''') &= -4\pi\int d\mathbf{r}'\,\nabla''\,\nabla'''\left(\nabla''\frac{1}{|\mathbf{r}''-\mathbf{r}'|}\cdot\nabla'''\frac{1}{|\mathbf{r}'-\mathbf{r}'''|}\right)\\
&= -4\pi\int d\mathbf{r}'\,\nabla''\,\nabla'''\left(\nabla'\frac{1}{|\mathbf{r}''-\mathbf{r}'|}\cdot\nabla'\frac{1}{|\mathbf{r}'-\mathbf{r}'''|}\right)\\
&= -4\pi\int d\mathbf{r}'\,\nabla''\,\nabla'''\left(\frac{1}{|\mathbf{r}''-\mathbf{r}'|}\,\Delta'\frac{1}{|\mathbf{r}'-\mathbf{r}'''|}\right)\\
&= -(4\pi)^2\,\nabla''\,\nabla'''\frac{1}{|\mathbf{r}''-\mathbf{r}'''|}\\
&= -4\pi\,\Delta''\mathbf{F}(\mathbf{r}''-\mathbf{r}''')
\end{aligned}$$

Since both $\mathbf{F}(\mathbf{r})$ and $\mathbf{G}(\mathbf{r})$ vanish as $r\to\infty$ it follows that

$$\mathbf{G}(\mathbf{r}) = -4\pi F(\mathbf{r})$$

and hence

$$I_3 = -4\pi I_2$$

Thus collecting terms we finally get

$$E_{\mathrm{H}} = \frac{1}{4}\int d\mathbf{r}'\int d\mathbf{r}''\left\{\frac{\mathbf{j}(\mathbf{r}')\cdot\mathbf{j}(\mathbf{r}'')}{|\mathbf{r}'-\mathbf{r}''|} + \frac{\mathbf{j}(\mathbf{r}')\cdot(\mathbf{r}'-\mathbf{r}'')\,\mathbf{j}(\mathbf{r}')\cdot(\mathbf{r}'-\mathbf{r}'')}{|\mathbf{r}'-\mathbf{r}''|^3}\right\} \tag{9.3}$$

This expression applied to electrons is known as Breit's Hamiltonian.

For moving discrete charges, $\mathbf{j}\approx\dfrac{\mathbf{v}}{c}\varrho$ so that

$$\frac{E_{\mathrm{H}}}{E_{\mathrm{E}}}\sim\frac{v^2}{c^2}$$

and hence for slow electrons, the main part of the interaction is electrostatic.

10 LIENARD–WIECHERT POINT POTENTIALS

In Section 7 we saw how φ and ψ can be obtained from ϱ and $\mathbf{j}$. Suppose the charge is concentrated in a domain of dimension d, small compared to all other distances involved in the problem, and let the charge move along a certain trajectory, $\mathbf{r}_0(t)$. For such a case, the method developed so far is clumsy, as integrations over all space are taken when the only part which gives anything is concentrated in a small volume. We can represent the charge and current densities by means of delta functions, thus

$$\varrho(\mathbf{r}, t) = e\,\delta(\mathbf{r} - \mathbf{r}_0(t))$$

$$\mathbf{j}(\mathbf{r}, t) = e\frac{\mathbf{v}(t)}{c}\delta(\mathbf{r} - \mathbf{r}_0(t))$$

where $\mathbf{v}(t) = \dot{\mathbf{r}}_0(t)$, the dot denoting differentiation with respect to t. It is easily verified that this representation satisfies the charge conservation law:

$$\nabla \cdot \mathbf{j}(\mathbf{r}, t) + \frac{1}{c}\frac{\partial\varrho(\mathbf{r}, t)}{\partial t} = \frac{e}{c}[\mathbf{v}(t) \cdot \nabla\,\delta(\mathbf{r} - \mathbf{r}_0(t)) - \nabla\,\delta(\mathbf{r} - \mathbf{r}_0(t)) \cdot \dot{\mathbf{r}}_0(t)]$$

$$= 0$$

Substituting the expression for ϱ into (7.7) we obtain

$$\varphi(\mathbf{r}, t) = e\int\frac{\delta\left(\mathbf{r}' - \mathbf{r}_0\left(t - \dfrac{|\mathbf{r} - \mathbf{r}'|}{c}\right)\right)}{|\mathbf{r} - \mathbf{r}'|}\,d\mathbf{r}'$$

The integrand contributes to the integral only at

$$\mathbf{r}' = \mathbf{r}_0(t_{ret}(\mathbf{r}, t))$$

where t_{ret} is the "retarded time," defined implicitly by the equation

$$t_{ret}(\mathbf{r}, t) = t - \frac{|\mathbf{r} - \mathbf{r}_0(t_{ret}(\mathbf{r}, t))|}{c}$$

Therefore we may write

$$\varphi(\mathbf{r}, t) = \frac{e}{|\mathbf{r} - \mathbf{r}_0(t_{ret}(\mathbf{r}, t))|}\int\delta(\mathbf{r}'')\frac{\partial(\mathbf{r}')}{\partial(\mathbf{r}'')}\,d\mathbf{r}''$$

where

$$\mathbf{r}'' = \mathbf{r}' - \mathbf{r}_0\left(t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)$$

In order to calculate the Jacobian $\partial(\mathbf{r}')/\partial(\mathbf{r}'')$ we first compute the tensor

$$\nabla'\,\mathbf{r}'' = \mathbf{1} - \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}\,\frac{\mathbf{v}\left(t - \dfrac{|\mathbf{r} - \mathbf{r}'|}{c}\right)}{c}$$

and then take its determinant. The result is

$$\frac{\partial(\mathbf{r}'')}{\partial(\mathbf{r}')} = 1 - \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \cdot \frac{\mathbf{v}\left(t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)}{c}$$

whence finally

$$\varphi = \left[\frac{e}{R\left(1 - \frac{v_R}{c}\right)}\right]_{\text{ret}} \tag{10.1}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$ and v_R is the component of velocity in the direction of $\mathbf{R}$:

$$v_R = \frac{\mathbf{R}}{R} \cdot \mathbf{v}$$

In exactly the same way we get

$$\boldsymbol{\psi} = \left[\frac{e\frac{\mathbf{v}}{c}}{R\left(1 - \frac{v_R}{c}\right)}\right]_{\text{ret}} \tag{10.2}$$

(10.1) and (10.2) are called the Lienard–Wichert potentials. They are rigorous for idealized point charges. Their validity when the charge is spread over a domain of dimension d may be determined as follows. The retarded time for different points of the charge distribution now varies by an amount $\Delta t_{\text{ret}} \sim d/c$. Also R varies by an amount $\Delta R \sim d$. In order that these variations have negligible effect on (10.1) and (10.2) we must have

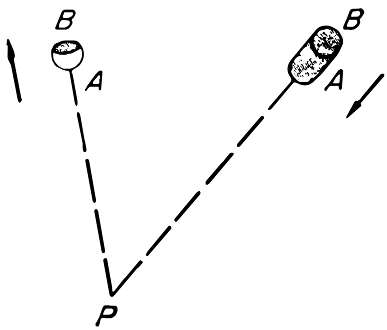
$$d \ll R, \quad \frac{|\dot{\mathbf{v}}|\, d}{c^2} \ll 1, \quad \frac{|\ddot{\mathbf{v}}|\, d^2}{c^3} \ll 1$$

etc.

As long as these conditions are satisfied, the above expressions for the potentials hold. It is not necessary that the charge be microscopic as in the case of electrons or nuclei; the formulae may be applied to macroscopic charges, such as a charged pith ball. For electrons we usually consider

$$d \sim \frac{e^2}{mc^2}$$

We can see qualitatively why an approaching charge gives a larger field than a receding one at the same distance. The retarded time at A is later than that at B and so for the approaching charge, A is effectively

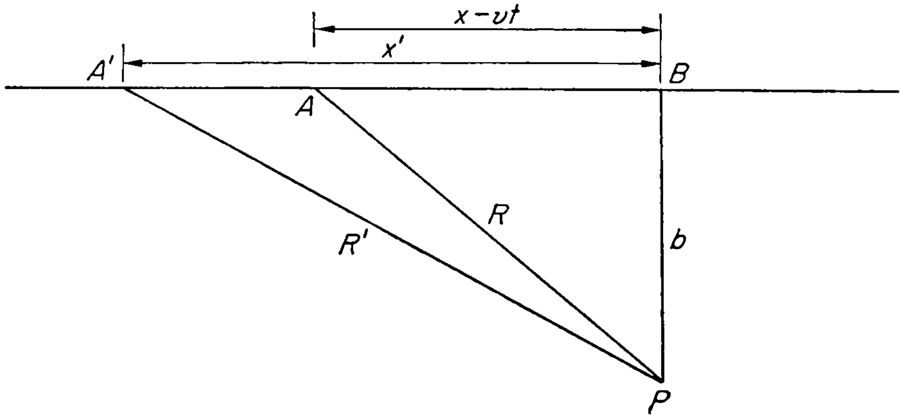


further away from B, while for the receding charge, A is effectively nearer B than the actual distance. This means that the approaching charge has the effect of a larger charge, and the receding charge a smaller one. Thus the former gives a larger field at P than the latter.

If we apply this idea to nebulae, we must have more nebulae to give the observed intensity when they are receding than when they are at rest with respect to the earth.

11 FIELD OF A UNIFORMLY MOVING POINT CHARGE

Let a charge e move with uniform velocity, v, along the x axis. We want to find the field produced by it at a point P at a perpendicular distance, b,



from the line of motion. Let us calculate the field at P at a time t when the charge is at A. We shall denote retarded quantities by a prime. Then

$$R'^2 = c^2(t - t')^2 = x'^2 + b^2 = v^2(t_0 - t')^2 + b^2 \qquad (11.1)$$

where t_0 is the time when the charge reaches B, and

$$\frac{v'_R}{c} = \frac{x'}{R'}\frac{v}{c} \tag{11.2}$$

By means of these equations, we can calculate the Lienard–Wiechert potentials in terms of instantaneous quantities. From (11.1) we have

$$c^2(t - t')^2 = v^2(t' - t_0)^2 + b^2$$

The solution of this quadratic for t' gives

$$t' = \frac{c^2 t - v^2 t_0 \pm \sqrt{c^2 v^2 (t_0 - t)^2 + (c^2 - v^2)\, b^2}}{(c^2 - v^2)}$$

where we must take the negative sign to obtain the retarded time. This gives

$$\frac{x'}{v} = \frac{c^2(t_0 - t) + \sqrt{c^2 v^2 (t_0 - t)^2 + (c^2 - v^2)\, b^2}}{(c^2 - v^2)}$$

and

$$\frac{R'}{c} = \frac{v^2(t_0 - t) + \sqrt{c^2 v^2 (t_0 - t)^2 + (c^2 - v^2)\, b^2}}{(c^2 - v^2)}$$

Hence

$$R' - \frac{v}{c}x' = R^* = \sqrt{v^2(t_0 - t)^2 + (1 - v^2/c^2)\, b^2} \tag{11.3}$$

and

$$\varphi = \left(\frac{e}{R\left(1 - \frac{v_R}{c}\right)}\right)_{\mathrm{ret}}$$

$$= \frac{e}{R' - \frac{v}{c}x'}$$

$$= \frac{e}{R^*} \tag{11.4}$$

$$\psi = \frac{e\mathbf{v}}{cR^*} \tag{11.5}$$

If x, y, z are the Cartesian coordinates of P, then

$$x = vt_0$$

$$y^2 + z^2 = b^2$$

if we choose the origin of time so that the charge is at the origin when $t = 0$. This gives

$$R^* = \sqrt{(x - vt)^2 + (1 - v^2/c^2)(y^2 + z^2)}$$

We can now obtain the fields by simple differentiation.

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \boldsymbol{\psi}}{\partial t} - \boldsymbol{\nabla}\varphi$$

$$= -\frac{e\mathbf{v}}{R^{*3}}\frac{v}{c^2}(x - vt) + \frac{e}{R^{*3}}\{(x - vt)\,a_x$$

$$+\left(1 - \frac{v^2}{c^2}\right)(ya_y + za_z)\}$$

where a_x, a_y, a_z are unit vectors along the x, y, z directions respectively. Thus

$$\mathbf{E} = \frac{e\mathbf{R}}{R^{*3}}\left(1 - \frac{v^2}{c^2}\right) \tag{11.6}$$

Next

$$\mathbf{H} = \boldsymbol{\nabla} \times \boldsymbol{\psi}$$

$$= -\left[\frac{\mathbf{v}}{c} \times \boldsymbol{\nabla}\frac{e}{R^*}\right]$$

$$= \frac{\mathbf{v}}{c} \times \mathbf{E} \tag{11.7}$$

since $\frac{1}{c}\frac{\partial \boldsymbol{\psi}}{\partial t}$ and $\mathbf{v}$ are parallel. We see that though the radiation is sent out from the retarded position, due to interference, the lines of electric force point from the instantaneous position of the charge. We also see that the field is not spherically symmetric as in the static case, but is stronger in the direction perpendicular to the line of motion than along it. In other words, the lines of force are more dense in the equatorial plane than in the polar directions. In the equatorial plane

$$|\mathbf{E}| = \frac{e}{\sqrt{1 - v^2/c^2}\,(y^2 + z^2)}$$

and in the polar direction (at $t = 0$)

$$|\mathbf{E}| = \frac{e}{x^2}\left(1 - \frac{v^2}{c^2}\right)$$

Thus vor v very close to c, almost all of the electric field is in the equatorial plane, and **E**, **H** and **v** will be perpendicular to each other so that the effect of the passage of a charge close by has practically the same effect as a pulse of radiation. We shall apply this idea to find how the ionizing power of particles varies with the velocity. Consider the case where a charged particle such as an α-particle or an electron passes by an atom. Most of the effect will be due to the transverse electric field and, of this, only the Fourier component which is in resonance with the atomic electron will be effective. Thus, taking the Fourier component, we have

$$\mathrm{E}_{\perp}^{\nu} = \int E_{\perp}\, e^{-i\nu t}\, dt$$

$$= e(1 - \beta^2)\, b \int_{-\infty}^{+\infty} \frac{e^{-i\nu t}\, dt}{\{(x - vt)^2 + (1 - \beta^2)\, b^2\}^{3/2}}$$

where $\beta = v/c$ and b is the distance of the atom from the line of motion of the ionizing particle. Let $\tau = t - x/v$. Then

$$E_{\perp}^{\nu} = e(1 - \beta^2)\, b\, e^{-i\frac{x}{v}\nu} \int_{-\infty}^{+\infty} \frac{e^{-i\nu t}\, d\tau}{\{v^2\tau^2 + (1 - \beta^2)\, b^2\}^{3/2}}$$

$$= -e\, e^{-i\frac{x}{v}\nu} \frac{\partial}{\partial b} \int_{-\infty}^{+\infty} \frac{e^{-i\nu\tau}\, d\tau}{\{v^2\tau^2 + (1 - \beta^2)\, b^2\}^{1/2}}$$

Let

$$\frac{v\tau}{\sqrt{1 - \beta^2}\, b} = \sigma \qquad \frac{\sqrt{1 - \beta^2}\, b\nu}{v} = l$$

Then

$$E_{\perp}^{\nu} = -\frac{e}{v} e^{-\iota \frac{x}{v} \nu} \frac{\partial}{\partial b} \int_{-\infty}^{+\infty} \frac{\cos l\sigma \, d\sigma}{\sqrt{\sigma^2 + 1}}$$

since the part with sin $l\sigma$ integrates to zero. Similarly, from

$$E_{\parallel}^{\nu} = e(1 - \beta^2) \int_{-\infty} \frac{(x - vt)\, e^{-\iota \nu t}\, dt}{\{(x - vt)^2 + (1 - \beta^2)\, b^2\}^{3/2}}$$

we get

$$E_{\parallel}^{\nu} = -\frac{e}{v}(1 - \beta^2) \frac{\partial}{\partial x} e^{-\iota \frac{x}{v} \nu} \int_{-\infty}^{+\infty} \frac{\cos l\sigma}{\sqrt{\sigma^2 + 1}} d\sigma$$

Let

$$\int_{-\infty}^{+\infty} \frac{\cos l\sigma \, d\sigma}{\sqrt{\sigma^2 + 1}} = F(l)$$

This is actually a constant times the Hankel function of zero order, and we have the asymptotic behaviour

$$F(l) \sim \frac{e^{-l}}{\sqrt{l}} \quad \text{for} \quad l \gg 1$$

$$F(l) \sim \ln l \quad \text{for} \quad l \ll 1$$

and hence

$$|E_{\perp}^{\nu}|, |E_{\parallel}^{\nu}| \sim \frac{e^{-l}}{\sqrt{l}} \quad \text{for} \quad b \gg \frac{v}{\nu\sqrt{1 - \beta^2}}$$

$$\left.\begin{aligned} |E_{\perp}^{\nu}| &\sim \frac{1}{l} \\ |E_{\parallel}^{\nu}| &\sim \ln l \end{aligned}\right\} \quad \text{for} \quad b \ll \frac{v}{\nu\sqrt{1 - \beta^2}}$$

Thus we see that most of the ionization occurs in the region $l < 1$, and we may take $l = 1$ as the range within which the ionizations occur. Taking $\nu = 10^{15}$ Hz. we have, for typical α-particles,

$$\frac{v}{\nu\sqrt{1 - \beta^2}} \sim \frac{10^9}{10^{15}} \sim 10^{-6}\ \text{cm}$$

while for electrons of 10^{10} MeV

$$\sqrt{1-\beta^2} = \frac{m_0 c^2}{E} \approx \frac{5 \times 10^5}{10^{10}} = \frac{1}{2 \times 10^4}$$

and

$$\frac{v}{\nu\sqrt{1-\beta^2}} \sim \frac{3 \times 10^{10} \times 2 \times 10^4}{10^{15}} \sim 6 \text{ mm}$$

This is a very large value, but it is thought that electrons of such high energies do not occur and that most of these high energy particles are mesons. If this is the case, the range would be reduced by a factor of about 200.

We shall now give a simpler derivation of the potentials of a uniformly moving charged particle. To do this, we go back to the original differential equations for the potentials

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -4\pi\varrho \tag{11.8}$$

$$\Delta\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi\varrho\frac{\mathbf{v}}{c} \tag{11.9}$$

and make a transformation to a coordinate system (x', y', z') in which ϱ is static. Let us again consider the particle to be moving along the x-axis. If

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t$$

then

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} = \frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}$$

and since if ϱ is static in the primed system, φ will also be static in this system, (11.8) becomes

$$\Delta'\varphi - \frac{v^2}{c^2}\frac{\partial^2\varphi}{\partial x'^2} = -4\pi\varrho$$

or

$$\left\{(1-\beta^2)\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right\}\varphi = -4\pi\varrho$$

Let

$$x'' = \frac{x'}{\sqrt{1-\beta^2}}, \quad y'' = y', \quad z'' = z'$$

Then

$$\left\{\frac{\partial^2}{\partial x''^2} + \frac{\partial^2}{\partial y''^2} + \frac{\partial^2}{\partial z''^2}\right\}\varphi = -\frac{4\pi\varrho''}{\sqrt{1-\beta^2}} \tag{11.10}$$

where

$$\varrho'' = \sqrt{1-\beta^2}\,\varrho$$

is the charge density in the new coordinates x'', y'', z''. Similarly, (11.9) can be transformed into

$$\left\{\frac{\partial^2}{\partial x''^2} + \frac{\partial^2}{\partial y''^2} + \frac{\partial^2}{\partial z''^2}\right\}\boldsymbol{\psi} = -\frac{4\pi\varrho''}{\sqrt{1-\beta^2}}\,\frac{\mathbf{v}}{c} \tag{11.11}$$

For a point charge, the solutions of these equations are, for the case of charge e at the origin

$$\varphi = \frac{e}{\sqrt{1-\beta^2}\sqrt{x''^2+y''^2+z''^2}}$$

$$= \frac{e}{\sqrt{(x-vt)^2+(1-\beta^2)(y^2+z^2)}}$$

$$\boldsymbol{\psi} = \frac{\mathbf{v}}{c}\varphi$$

and these are exactly the same as (11.4) and (11.5).

The transformation we used above is actually a part of the Lorentz transformation

$$x'' = \frac{x-vt}{\sqrt{1-\beta^2}}, \quad y'' = y, \quad z'' = z, \quad t'' = \frac{t-\dfrac{v}{c^2}x}{\sqrt{1-\beta^2}}$$

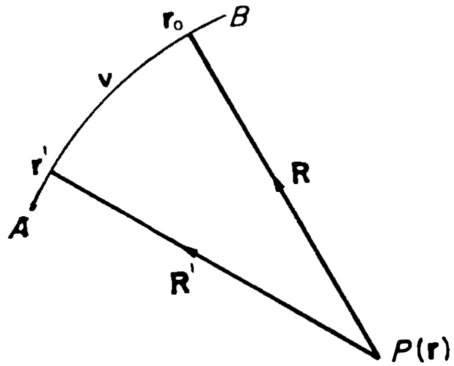
We did not use the transformation of the time but used instead

$$\varrho'' = \frac{\varrho - \dfrac{v}{c}j}{\sqrt{1-\beta^2}} = \sqrt{1-\beta^2}\,\varrho \quad \left(\text{since } j = \frac{v}{c}\varrho\right)$$

We shall consider these transformations more fully when we discuss the special theory of relativity.

12 FIELD OF AN ACCELERATED POINT CHARGE

Suppose a charged particle describes the trajectory *AB*. We want to find the field due to it at point $P(\mathbf{r})$ at time t.



Let the charge be at $\mathbf{r}_0$ at time t and at $\mathbf{r}'$ at the retarded time $t' = t - R'/c$. Also let $\mathbf{n} = \mathbf{R}/R$, and $\boldsymbol{\beta} = \mathbf{v}/c$. Then

$$\varphi = \left[\frac{e}{R(1 + \boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\text{ret}}$$

$$\boldsymbol{\psi} = \left[\frac{e\boldsymbol{\beta}}{R(1 + \boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\text{ret}}$$

and

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial\boldsymbol{\psi}}{\partial t}, \quad \mathbf{H} = \nabla\times\boldsymbol{\psi}$$

These differentiations are made complicated by the fact that the potentials are retarded. We shall denote by primes all retarded quantities. We have

$$\nabla\varphi = \nabla_{t'=\text{const}}\varphi + \frac{\partial\varphi}{\partial t'}\nabla t'$$

and

$$\nabla t' = -\frac{1}{c}\nabla_{t'=\text{const}}R' - \frac{1}{c}\frac{\partial R'}{\partial t'}\nabla t'$$

Now

$$\nabla_{t'=\text{const}}R' = -\frac{\mathbf{R}'}{R} = -\mathbf{n}'$$

$$\frac{\partial R'}{\partial t'} = c\boldsymbol{\beta}'\cdot\mathbf{n}'$$

Therefore

$$\nabla t' = \frac{\mathbf{n}'}{c(1 + \boldsymbol{\beta}'\cdot\mathbf{n}')}$$

Further

$$\nabla_{t'=\text{const}}\varphi = \nabla_{t'=\text{const}} \frac{e}{R' + \boldsymbol{\beta}' \cdot \mathbf{R}'}$$

$$= + \frac{e(\mathbf{n}' + \boldsymbol{\beta}')}{(R' + \boldsymbol{\beta}' \cdot \mathbf{R}')^2}$$

$$\frac{\partial \varphi}{\partial t'} = - \frac{e}{(R' + \boldsymbol{\beta}' \cdot \mathbf{R}')^2} \{c\boldsymbol{\beta}' \cdot \mathbf{n}' + \dot{\boldsymbol{\beta}} \cdot \mathbf{R}' + c\beta'^2\}$$

where a dot denotes differentiation with respect to t'. Thus

$$\nabla \varphi = + \frac{e(\mathbf{n}' + \boldsymbol{\beta}')}{(1 + \boldsymbol{\beta}' \cdot \mathbf{n}')^2 R'^2} - \frac{e\mathbf{n}'(\boldsymbol{\beta}' \cdot \mathbf{n}' + (\dot{\boldsymbol{\beta}}'/c) \cdot \mathbf{R}' + \beta'^2)}{(1 + \boldsymbol{\beta}' \cdot \mathbf{n}')^3 R'^2}$$

Also

$$\frac{\partial \boldsymbol{\psi}}{\partial t} = \frac{\partial \boldsymbol{\psi}}{\partial t'} \frac{\partial t'}{\partial t}$$

But

$$\frac{\partial t'}{\partial t} = 1 - \frac{1}{c} \frac{\partial R'}{\partial t'} \frac{\partial t'}{\partial t}$$

which yields

$$\frac{\partial t'}{\partial t} = \frac{1}{1 + \boldsymbol{\beta}' \cdot \mathbf{n}'}$$

and since

$$\frac{\partial \boldsymbol{\psi}}{\partial t'} = \boldsymbol{\beta}' \frac{\partial \varphi}{\partial t'} + \varphi \dot{\boldsymbol{\beta}}'$$

we have

$$\frac{1}{c} \frac{\partial \boldsymbol{\psi}}{\partial t} = \frac{1}{(1 + \boldsymbol{\beta}' \cdot \mathbf{n}')} \left\{ - \frac{e\boldsymbol{\beta}'}{(R' + \boldsymbol{\beta}' \cdot \mathbf{R}')^2} \left(\boldsymbol{\beta}' \cdot \mathbf{n}' + \frac{\dot{\boldsymbol{\beta}}'}{c} \cdot \mathbf{R}' + \beta'^2 \right) + \frac{e\dot{\boldsymbol{\beta}}'/c}{R' + \boldsymbol{\beta}' \cdot \mathbf{R}'} \right\}$$

Thus, finally, on collecting the terms

$$\mathbf{E} = \left[-e \left\{ \frac{\mathbf{n} + \boldsymbol{\beta}}{(1 + \boldsymbol{\beta} \cdot \mathbf{n})^2 R^2} - \frac{\mathbf{n}(\boldsymbol{\beta} \cdot \mathbf{n} + \beta^2)}{(1 + \boldsymbol{\beta} \cdot \mathbf{n})^3 R^2} - \frac{\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{n} + \beta^2)}{(1 + \boldsymbol{\beta} \cdot \mathbf{n})^3 R^2} - \frac{(\mathbf{n} + \boldsymbol{\beta})\, \mathbf{n} \cdot \dot{\boldsymbol{\beta}}}{(1 + \boldsymbol{\beta} \cdot \mathbf{n})^3 cR} + \frac{\dot{\boldsymbol{\beta}}(1 + \boldsymbol{\beta} \cdot \mathbf{n})}{(1 + \boldsymbol{\beta} \cdot \mathbf{n})^3 cR} \right\} \right]_{\text{ret}}$$

$$= \left[-e \left\{ \frac{(1 - \beta^2)\,(\mathbf{n} + \boldsymbol{\beta})}{(1 + \mathbf{n} \cdot \boldsymbol{\beta})^3 R^2} + \frac{\dot{\boldsymbol{\beta}}(1 + \boldsymbol{\beta} \cdot \mathbf{n}) - \mathbf{n} \cdot \dot{\boldsymbol{\beta}}(\mathbf{n} + \boldsymbol{\beta})}{(1 + \mathbf{n} \cdot \boldsymbol{\beta})^3 cR} \right\} \right]_{\text{ret}}$$

Now to find **H** we have

$$\nabla \times \boldsymbol{\psi} = \nabla_{t'} \times \boldsymbol{\psi} + \nabla t' \times \frac{\partial \boldsymbol{\psi}}{\partial t'}$$

where the subscript t' means that t' is to be kept constant for this differentiation.

Now

$$\nabla_{t'} \times (\boldsymbol{\beta}'\varphi) = -\boldsymbol{\beta}' \times \nabla'_{t'}\varphi$$

$$= \frac{e\mathbf{n}' \times \boldsymbol{\beta}'}{(1 + \boldsymbol{\beta}' \cdot \mathbf{n}')^2 R'^2}$$

$$\left[\nabla t' \times \frac{\partial \boldsymbol{\psi}}{\partial t'}\right] = \frac{\mathbf{n}' \times \boldsymbol{\beta}'}{c(1 + \boldsymbol{\beta}' \cdot \mathbf{n}')} \frac{\partial \varphi}{\partial t'} + \frac{\mathbf{n}' \times \dot{\boldsymbol{\beta}}'\varphi}{c(1 + \boldsymbol{\beta}' \cdot \mathbf{n}')}$$

$$= -\frac{e\mathbf{n}' \times \boldsymbol{\beta}'}{(1 + \boldsymbol{\beta}' \cdot \mathbf{n}')^3 R'^2}\left(\boldsymbol{\beta}' \cdot \mathbf{n}' + \frac{\dot{\boldsymbol{\beta}}' \cdot \mathbf{R}'}{c} + \beta'^2\right) + \frac{e\mathbf{n}' \times \dot{\boldsymbol{\beta}}'}{c(1 + \boldsymbol{\beta}' \cdot \mathbf{n}')^2 R'}$$

Therefore

$$\mathbf{H} = \left[e\left\{\frac{[\mathbf{n} \times \boldsymbol{\beta}](1 - \beta^2)}{(1 + \boldsymbol{\beta} \cdot \mathbf{n})^3 R^2} + \frac{[\mathbf{n} \times \dot{\boldsymbol{\beta}}](1 + \boldsymbol{\beta} \cdot \mathbf{n}) - [\mathbf{n} \times \boldsymbol{\beta}]\,\mathbf{n} \cdot \dot{\boldsymbol{\beta}}}{(1 + \mathbf{n} \cdot \boldsymbol{\beta})^3 cR}\right\}\right]_{\mathrm{ret}}$$

and comparing this with **E** we see that

$$\mathbf{H} = -[\mathbf{n} \times \mathbf{E}]_{\mathrm{ret}}$$

Thus **H** is always perpendicular to **E** and to $\mathbf{n}_{\mathrm{ret}}$. The first part of **E** is nearly parallel to $\mathbf{n}'$ for ordinary velocities, and the second part is exactly perpendicular to $\mathbf{n}'$ which can be seen immediately by taking a scalar product with $\mathbf{n}'$. The first term, which falls off as $1/R'^2$, gives the quasi-static field, while the second falls off as $1/R'$ and gives the wave zone field. The ratio of the magnitudes of the two terms is

$$\frac{\dot{\mathbf{v}}'R'}{c^2}$$

and this has a simple meaning since

$$t - t' = \frac{|\mathbf{r} - \mathbf{r}'|}{c} = \frac{1}{c}\left|\mathbf{r} - \mathbf{r}_0 + \mathbf{v}'(t - t') + \frac{\dot{\mathbf{v}}'}{2}(t - t')^2 + \cdots\right|$$

and the condition that this series converge rapidly is

$$\frac{\dot{\mathbf{v}}' R'}{c^2} \ll 1$$

Thus if this condition holds, we have

$$\mathbf{E} = \left[-e \left\{ \frac{(1 - \beta^2)(\mathbf{n} + \boldsymbol{\beta})}{(1 + \mathbf{n} \cdot \boldsymbol{\beta})^3 R^2} \right\} \right]_{t' = t - \frac{|\mathbf{r} - \mathbf{r}_0 + \mathbf{v}'(t - t')|}{c}}$$

We shall show that this expression is the same as (11.6) for the field of a uniformly moving charge. To the approximation we are making, $\boldsymbol{\beta}$ is constant, and we have

$$R'(\boldsymbol{\beta} + \mathbf{n}') = \mathbf{R} = R\mathbf{n}$$

Thus our proof will be complete if we can show that

$$(1 + \mathbf{n}' \cdot \boldsymbol{\beta})^3 R'^3 = R^{*3}$$

From the above relation we get

$$R'(\beta^2 + \mathbf{n}' \cdot \boldsymbol{\beta}) = R\mathbf{n} \cdot \boldsymbol{\beta}$$

$$R'(1 + \mathbf{n}' \cdot \boldsymbol{\beta}) = R\mathbf{n} \cdot \boldsymbol{\beta} + R'(1 - \beta^2)$$

Now

$$R^* = \left\{ \frac{(\mathbf{R} \cdot \boldsymbol{\beta})^2}{\beta^2} + \left[\mathbf{R} - \left(\frac{\mathbf{R} \cdot \boldsymbol{\beta}}{\beta^2} \right) \boldsymbol{\beta} \right]^2 (1 - \beta^2) \right\}^{1/2}$$

$$= R\sqrt{1 - \beta^2 + (\mathbf{n} \cdot \boldsymbol{\beta})^2}$$

and moreover

$$t - t' = \left| \frac{\mathbf{R}}{c} - \boldsymbol{\beta}(t - t') \right|$$

which may be solved to yield

$$t - t' = \frac{R}{c} \, \frac{-\mathbf{n} \cdot \boldsymbol{\beta} + \sqrt{(\mathbf{n} \cdot \boldsymbol{\beta})^2 + (1 - \beta^2)}}{1 - \beta^2}$$

Therefore

$$R' = (t - t')\, c = \frac{-R(\mathbf{n} \cdot \boldsymbol{\beta}) + R^*}{1 - \beta^2}$$

Hence

$$R'(1 + \mathbf{n}' \cdot \boldsymbol{\beta}) = R^*$$

and we have shown that

$$\mathbf{E} = \frac{e\mathbf{R}(1 - \beta^2)}{R^{*3}}$$

We have seen that for the general case

$$\mathbf{H} = -\mathbf{n}' \times \mathbf{E}$$

This reduces to the expression

$$\mathbf{H} = \boldsymbol{\beta} \times \mathbf{E}$$

when the velocity is constant. The proof is left as an exercise.

EXERCISE 15 Show that for the case of uniform motion

$$\mathbf{n}' \times \mathbf{R} = -\boldsymbol{\beta} \times \mathbf{R}$$

13 RATE OF RADIATION OF ENERGY FROM AN ACCELERATED POINT CHARGE

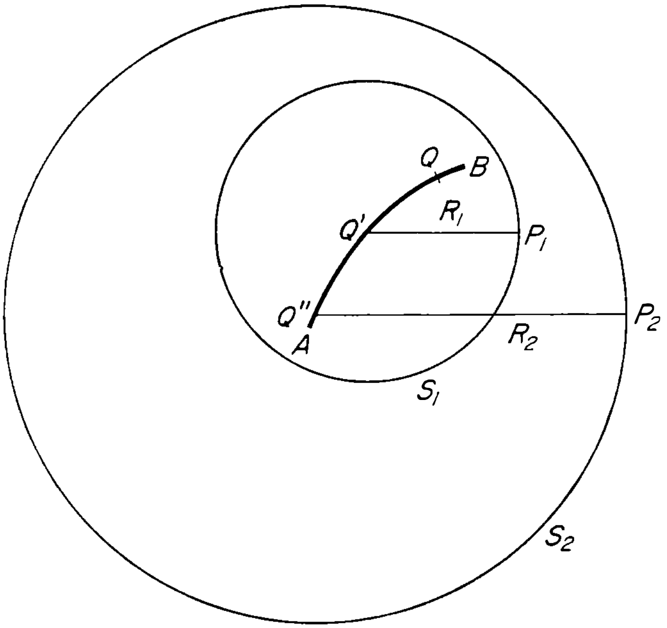
We have seen that the general expression of $\mathbf{E}$ for a charged particle can be considered as a sum of two terms: The first, which is the same as that for a uniform velocity, and the second, which depends on the acceleration. For the latter part, $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{h}$ are perpendicular to each other, and $|\mathbf{E}| = |\mathbf{H}|$, and this is the field of the electromagnetic waves radiated by the particle. Thus

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H} = -\frac{c\mathbf{n}}{4\pi}E^2$$

$$\mathbf{S} = \left[-\frac{\mathbf{n}e^2}{4\pi c^3}\left\{\frac{\dot{v}^2(1 + \boldsymbol{\beta}\cdot\mathbf{n})^2 - 2(\mathbf{n}\cdot\dot{\mathbf{v}})(1 + \boldsymbol{\beta}\cdot\mathbf{n})(\mathbf{n}\cdot\dot{\mathbf{v}} + \boldsymbol{\beta}\cdot\dot{\mathbf{v}})}{(1 + \mathbf{n}\cdot\boldsymbol{\beta})^6 R^2} + \frac{(\mathbf{n}\cdot\dot{\mathbf{v}})^2(1 + 2\mathbf{n}\cdot\boldsymbol{\beta} + \beta^2)}{(1 + \mathbf{n}\cdot\boldsymbol{\beta})^6 R^2}\right\}\right]_{\text{ret.}}$$

$$= \left[-\frac{\mathbf{n}e^2}{4\pi c^3}\left\{\frac{\dot{\mathbf{v}}^2(1 + \boldsymbol{\beta}\cdot\mathbf{n})^2 - 2(\mathbf{n}\cdot\dot{\mathbf{v}})(1 + \boldsymbol{\beta}\cdot\mathbf{n})(\boldsymbol{\beta}\cdot\dot{\mathbf{v}}) - (\mathbf{n}\cdot\dot{\mathbf{v}})^2(1 - \beta^2)}{(1 + \mathbf{n}\cdot\boldsymbol{\beta})^6 R^2}\right\}\right]_{\text{ret.}}$$

At first sight it seems that the total rate of radiation may be obtained by integrating S over a sphere with the center at the retarded position of the charge. However, this procedure does not give the correct result. To see why this is so, let us look at the problem in more detail. Suppose AB is the trajectory of the particle. At time t, let its position be Q. The radiation which arrives at P_1 at time t_1 came from the retarded position Q' and the radiation emitted at Q' lies on the surface of the sphere, S_1, with center at Q' and radius $R_1 = Q'P_1$. If we consider another point, P_2, the radiation there at time t came from Q'' and the radiation emitted at Q'' lies on the



surface of the sphere, S_2, with center at Q'' and radius $R_2 = Q''P_2$. It is clear that in general the energy flowing across S_1 will not be equal to that flowing across S_2. This is why we cannot obtain the rate of radiation by considering the flow of S energy across a fixed sphere. The correct method is to consider the spheres S_1 and S_2 to move outward with the radiation at velocity c. Then the energy of radiation enclosed within S_1 and S_2 will be constant, and the rate of radiation is this energy divided by the time taken to radiate it. Now the energy within S_1 and S_2, when their radii differ by dR, is

$$dE = R^2\, dR \int d\Omega\, W(1 + \boldsymbol{\beta} \cdot \mathbf{n})$$

$$\frac{dE}{dt} = c \int R^2\, d\Omega (1 + \boldsymbol{\beta} \cdot \mathbf{n})\, W$$

$$= \int R^2\, d\Omega\, \mathbf{S} \cdot (-\mathbf{n})\, (1 + \boldsymbol{\beta} \cdot \mathbf{n})$$

Since for $|\mathbf{E}| = |\mathbf{H}|$

$$W = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{H}^2) = \frac{(-\mathbf{n}) \cdot \mathbf{S}}{c}$$

Putting in the value for **S**, we have

$$\frac{dE}{dt} = \frac{e^2}{4\pi c^3}\int d\Omega \frac{\{\dot{\mathbf{v}}^2(1+\boldsymbol{\beta}\cdot\mathbf{n})^2 - 2(\mathbf{n}\cdot\dot{\mathbf{v}})(1+\boldsymbol{\beta}\cdot\mathbf{n})(\boldsymbol{\beta}\cdot\dot{\mathbf{v}}) - (\mathbf{n}\cdot\dot{\mathbf{v}})^2(1-\beta^2)\}}{(1+\mathbf{n}\cdot\boldsymbol{\beta})^5} \tag{13.1}$$

This is the correct expression, since it agrees with the formula derived from the method of special relativity.

EXERCISE 16 Show why the factor $(1 + \boldsymbol{\beta}\cdot\mathbf{n})$ has to be included in the integral over the angles of the rate of radiation.

Let us consider some special cases. First, if $\boldsymbol{\beta}$ is very small, then

$$\frac{dE}{dt} = \frac{e^2}{4\pi c^3}\int d\boldsymbol{\Omega}\,\{\dot{\mathbf{v}}^2 - (\mathbf{n}\cdot\mathbf{v})^2\}$$

If we take the polar axis along $\mathbf{n}$, then

$$\frac{dE}{dt} = \frac{e^2}{4\pi c^3}\int\int d\varphi\, d(\cos\theta)\,(\dot{\mathbf{v}}^2 - \cos^2\theta\,\dot{\mathbf{v}}^2)$$

$$= \frac{2e^2}{3c^3}\dot{\mathbf{v}}^2 \tag{13.2}$$

We see that the angular distribution is $\sin^2\theta$ so that it is a dipole radiation. Thus, the condition $\lambda \gg d$ for the validity of dipole radiation is equivalent to the condition $v \ll c$. For a charge, e, oscillating harmonically we have

$$\ddot{x} = -\omega^2 x$$
$$= -\omega^2 a \cos\omega t$$

so that

$$\frac{dE}{dt} = \frac{2\omega^4 e^2 a^2}{3c^3}\cos^2\omega t$$

$$\left(\frac{dE}{dt}\right)_{av} = \frac{\omega^4 e^2 a^2}{3c^3} \tag{13.3}$$

EXERCISE 17 If $\dfrac{\omega a}{c} = \dfrac{1}{3}$, then $v \sim c$ at the center of oscillation. Compare the average rates of radiation calculated from the exact formula (13.1) and the approximate one (13.3).

The total energy of the oscillator is

$$H = \frac{ma^2\omega^2}{2}$$

Hence the fraction of its energy radiated in one cycle is

$$\frac{T}{H}\frac{dE}{dt} = \frac{2}{ma^2\omega^2}\,\frac{2\pi}{\omega}\,\frac{\omega^4 e^2 a^2}{3c^3}$$

$$= \frac{4\pi e^2\omega}{3mc^3}$$

$$= \frac{8\pi^2 e^2 \nu}{3mc^3} \tag{13.4}$$

where $\nu = 1/T$. This fraction is of order 1 for

$$\nu \sim \frac{mc^3}{e^2}$$

for electrons

$$\nu \sim \frac{mc^3}{e^2} \sim \frac{1}{\hbar} mc^2 \frac{\hbar c}{e^2}$$

$$\sim \frac{1}{\hbar} \times \frac{1}{2} \times 137 \text{ MeV}$$

$$\sim \frac{70 \text{ MeV}}{\hbar}$$

Let $\nu_0 = \dfrac{3mc^3}{8\pi^2 e^2}$. Then for $\dfrac{\nu}{\nu_0} \ll 1$, the damping factor is negligible and the effect of radiation need not be considered. We see that for electrons this condition is usually satisfied.

Next consider the case $\mathbf{v}$ perpendicular to $\dot{\mathbf{v}}$. Let $\mathbf{v}$ be along the z axis and $\dot{\mathbf{v}}$ along the x axis. Then putting $\gamma = \cos\theta$, we have

$$\frac{dE}{dt} = \frac{e^2}{4\pi c^3}\int_0^{2\pi} d\varphi \int_{-1}^{+1} d\gamma\, \frac{\{\dot{\mathbf{v}}^2(1-\beta\gamma)^2 - \dot{\mathbf{v}}^2(1-\gamma^2)(1-\beta^2)\cos^2\varphi\}}{(1-\beta\gamma)^5} \tag{13.5}$$

We see that there is a great concentration of radiation in the forward direction; the ratio of energy radiated in the forward direction to that radiated backward is

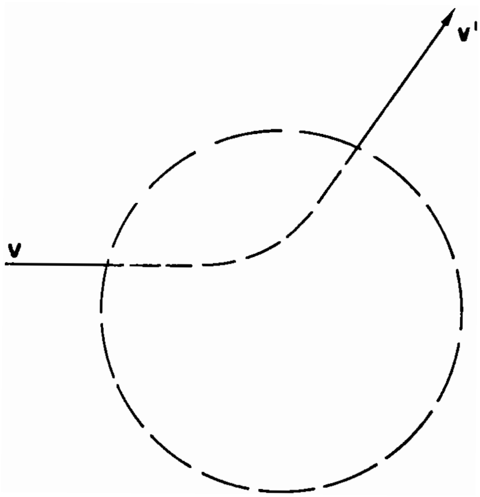
$$\left(\frac{1+\beta}{1-\beta}\right)^3$$

EXERCISE 18 Show that the rate of radiation from a charge e, moving in a circle of radius a, with constant angular velocity ω, is given by

$$\frac{dE}{dt} = \frac{2e^2}{3c^3}\,\frac{a^2\omega^4}{\left\{1 - \left(\dfrac{a\omega}{c^2}\right)^2\right\}}$$

14 APPLICATION TO A SIMPLE THEORY OF BREMSSTRAHLUNG

An electron moving with uniform velocity **v** comes near an atom and, while it is within the field of force, the electron is accelerated, and it then goes away with uniform velocity, **v**′. During the interval of time τ when



the electron is accelerated, it radiates electromagnetic waves, and we want to calculate the amount of energy it radiates. Now

$$\tau \sim \frac{a}{v}$$

where a is the size of the atom. From Thomas-Fermi model of the atom

$$a \approx \frac{1.3\hbar^2}{me^2Z^{1/3}} \sim 8 \text{ to } 2 \times 10^{-9} \text{ cm}$$

Since $v = 10^9$ cm/sec, $\tau \sim 10^{-18}$ sec. If $\nu\tau \ll 1$, where ν is the frequency of the radiated wave, we can derive a simple, general result. We shall deal only with this case. We can write

$$\begin{aligned}\nu\tau &\sim \frac{1.3\hbar^2\nu}{me^2Z^{1/3}v} \\ &= \frac{2h\nu}{mv^2} \times \frac{1.3}{4\pi Z^{1/3}} \times \frac{\hbar c}{e^2}\,\frac{v}{c} \\ &\approx \frac{2h\nu}{mv^2} \times \frac{14}{Z^{1/3}} \times \frac{v}{c}\end{aligned}$$

Now $\dfrac{2h\nu}{mv^2}$ is the fraction of the energy radiated in a single quantum. Hence, our condition is that the fraction of the energy radiated be small, or another way in which the condition may be satisfied is to have $v \ll c$. We shall assume this condition to hold. We shall only consider the wave zone field since this is the part which corresponds to the radiated wave. Making a Fourier analysis we have

$$\mathbf{E} = \int_{-\infty}^{+\infty} \mathbf{E}_\nu \, e^{2\pi i \nu t} \, d\nu$$

$$\mathbf{E}_\nu = \int_{-\infty}^{+\infty} \mathbf{E} e^{-2\pi i \nu t} \, dt$$

$$= \frac{e}{rc^2} \int_{-\infty}^{+\infty} dt \{\dot{\mathbf{v}} - \mathbf{n}(\mathbf{n} \cdot \dot{\mathbf{v}})\} e^{-2\pi i \nu t} \tag{14.1}$$

Since $\dot{\mathbf{v}}$ has non-zero value only in the short interval τ, we can write

$$\mathbf{E}_\nu = \frac{e}{rc^2} \{\Delta \mathbf{v} - \mathbf{n}(\mathbf{n} \cdot \Delta \mathbf{v})\}$$

when we take the collision to occur at $t = 0$. The total radiation, R, is given by

$$R = \int_{-\infty}^{+\infty} dt \int d\Omega \frac{cr^2}{4\pi} E^2$$

$$= \int d\Omega \frac{cr^2}{4\pi} \int_{-\infty}^{+\infty} dt \left\{ \int_{-\infty}^{+\infty} \mathbf{E}_\nu e^{2\pi i \nu t} \, d\nu \right\} \left\{ \int_{-\infty}^{+\infty} \mathbf{E}_{\nu'} e^{2\pi i \nu' t} \, d\nu' \right\}$$

The integral over t gives a delta function, and therefore, since $\mathbf{E}_{-\nu} = \bar{\mathbf{E}}_\nu$, we have

$$R = \int_0^\infty R_\nu \, d\nu$$

where

$$R_\nu = \int d\Omega \frac{cr^2}{2\pi} |\mathbf{E}_\nu|^2 \tag{14.2}$$

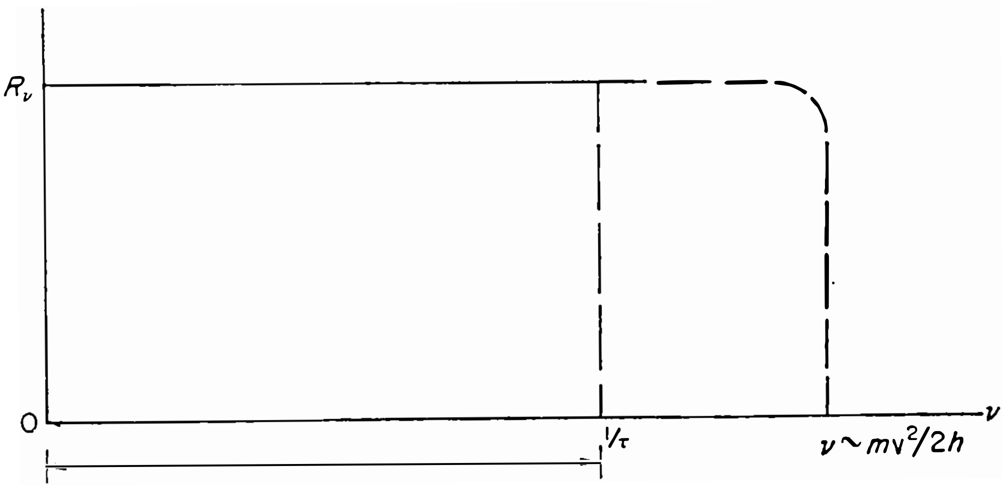
If we choose the polar axis along $\Delta\mathbf{v}$, we may write

$$|\Delta\mathbf{v} - \mathbf{n}(\mathbf{n}\cdot\Delta\mathbf{v})| = |\Delta\mathbf{v}|\sin\theta$$

$$R_\nu = \frac{e^2}{2\pi c^3}\int_0^{2\pi} d\varphi \int d(\cos\theta)\,|\Delta\mathbf{v}|^2\,(1 - \cos^2\theta)$$

$$= \frac{4e^2}{3c^3}\,|\Delta\mathbf{v}|^2 \tag{14.3}$$

Thus in the region of applicability, the intensity distribution function is constant, and since it must fall to zero for $\nu \sim \dfrac{mv^2}{2h}$, the intensity distribution curve has the following shape



The number of quanta emitted is given by

$$N_\nu\,d\nu = \frac{R_\nu}{h\nu}\,d\nu$$

$$= \frac{4e^2}{3hc}\left|\frac{\Delta\mathbf{v}}{c^2}\right|^2 \frac{d\nu}{\nu}$$

Thus a large number of long wave-length quanta come off.

Bremsstrahlung may be looked upon as due to "shaking off" of quanta from the field of an electron which is given a sudden jerk. The fields of the electron before and after the acceleration are different, and, if this change occurs in a time, τ, the Fourier components of the original field with $1/\nu \ll \tau$, cannot adjust themselves to the change within this time, and the difference in the components of the two fields comes off as radiation. The vector potential of the particle before collision is

$$\boldsymbol{\psi} = \frac{e\mathbf{v}}{cr}$$

remembering that we are making the assumption $\mathbf{v} \ll c$. After the collision, it is

$$\boldsymbol{\psi}' = \frac{e\mathbf{v}'}{cr}$$

We make the assumption that the collision time is very short compared to the period of the radiated wave. Then we can take r just before and just after the collision to be the same. Thus the change in the vector potential is

$$\Delta\boldsymbol{\psi} = \frac{e\,\Delta\mathbf{v}}{cr}$$

Making a space Fourier analysis of this, we have

$$\Delta\boldsymbol{\psi} = \sum_{\lambda}\int \boldsymbol{\varepsilon}_{\mathbf{k}\lambda} e^{2\pi i\mathbf{k}\cdot\mathbf{r}} \Delta\psi_{\mathbf{k}\lambda}\, d\mathbf{k}$$

where λ goes from 1 to 3 and

$$\Delta\psi_{\mathbf{k}\lambda} = \int (\Delta\boldsymbol{\psi}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\lambda})\, e^{-2\pi i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r}$$

$$= \frac{e(\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\lambda})}{c}\int_0^{\infty} dr \int_{-1}^{+1} d\mu \int_0^{2\pi} d\varphi\; r e^{-2\pi i\mathbf{k}r\mu}$$

$$= \frac{2e(\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\lambda})}{ck}\int_0^{\infty} \sin(2\pi kr)\, dr$$

$$= \frac{e(\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\lambda})}{\pi c k^2}\int_0^{\infty} \sin x\, dx \quad \text{where} \quad x = 2\pi kr$$

The value of the integral in this form is indeterminate, but we can use a damping factor e^{-ax} since

$$\int_0^\infty e^{-ax}\sin x\,dx = \frac{1}{a^2+1}$$

and take the limit $a \to 0$. We get

$$\int_0^\infty \sin x\,dx = 1$$

and hence

$$\Delta\psi_{\mathbf{k}\lambda} = \frac{e\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\lambda}}{\pi c k^2}$$

and

$$\Delta\boldsymbol{\psi} = \sum_\lambda \frac{e}{\pi c}\int \boldsymbol{\varepsilon}_{\mathbf{k}\lambda} e^{2\pi i\mathbf{k}\cdot\mathbf{r}}\frac{\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\lambda}}{k^2}\,d\mathbf{k}$$

The meaning of $\Delta\boldsymbol{\psi}$ can be understood better by considering the following analogy. A harmonic oscillator is undergoing forced oscillation with an amplitude A. Suddenly the magnitude of the external force is altered so that the amplitude of the forced oscillation is A'. The position of the oscillator at the instant when the change occurs does not correspond to that due to the new forced oscillation alone, and the difference is taken care of by the excitation of free oscillation of amplitude equal to the corresponding positions of the oscillator under the influence of the old and new forced oscillations at the instant when the change occurs. $\Delta\boldsymbol{\psi}$ corresponds to the amplitude of the free oscillation, and to obtain the vector potential of the radiation field, we must multiply each Fourier component by the corresponding time factor which we may conveniently take as

$$\cos(2\pi kct)$$

Thus

$$\boldsymbol{\psi} = \sum_\lambda \frac{e}{\pi c}\int \boldsymbol{\varepsilon}_{\mathbf{k}\lambda} e^{2\pi i\mathbf{k}\cdot\mathbf{r}}\frac{\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\lambda}}{k^2}\cos(2\pi kct)\,d\mathbf{k}$$

From this we can calculate the fields. Now we have seen in Section 7 that in the wave zone field, the contribution to $\mathbf{E}$ from φ is a component parallel to the propagation vector which just cancels off the parallel part coming from $\boldsymbol{\psi}$ and makes $\mathbf{E}$ perpendicular to $\mathbf{k}$. Thus we need not calculate this part; we can get $\mathbf{E}$ by taking just the part from $\boldsymbol{\psi}$ which is perpendicular to $\mathbf{k}$. Thus

$$\mathbf{E} = -\sum_{\lambda=1,2} \frac{e}{\pi c^2} \frac{\partial}{\partial t} \int \boldsymbol{\varepsilon}_{\mathbf{k}\lambda} e^{2\pi i \mathbf{k}\cdot\mathbf{r}} \frac{\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\lambda}}{k^2} \cos(2\pi kct)\, d\mathbf{k}$$

$$= \sum_{\lambda=1,2} \frac{2e}{c} \int \boldsymbol{\varepsilon}_{\mathbf{k}\lambda} e^{2\pi i \mathbf{k}\cdot\mathbf{r}} \frac{\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\lambda}}{k} \sin(2\pi kct)\, d\mathbf{k}$$

Next

$$\mathbf{H} = \nabla \times \sum_{\lambda=1,2,3} \frac{e}{\pi c} \int \boldsymbol{\varepsilon}_{\mathbf{k}\lambda} e^{2\pi i \mathbf{k}\cdot\mathbf{r}} \frac{\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\lambda}}{k^2} \cos(2\pi kct)\, d\mathbf{k}$$

$$= \sum_{\lambda=1,2} \frac{2ei}{c} \int [\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathbf{k}\lambda}]\, e^{2\pi i \mathbf{k}\cdot\mathbf{r}} \frac{\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\lambda}}{k^2} \cos(2\pi kct)\, d\mathbf{k}$$

Using the fact that

$$\sum_{\lambda=1,2} \boldsymbol{\varepsilon}_{\mathbf{k}\lambda}\boldsymbol{\varepsilon}_{\mathbf{k}\lambda} = \sum_{\lambda=1,2} \boldsymbol{\varepsilon}_{-\mathbf{k}\lambda}\boldsymbol{\varepsilon}_{-\mathbf{k}\lambda}$$

we now get

$$\int \mathbf{E}^2\, d\mathbf{r} = \sum_{\lambda=1,2} \cdot \sum_{\lambda'=1,2} \frac{4e^2}{c^2} \int d\mathbf{r} \int d\mathbf{k}'\, \boldsymbol{\varepsilon}_{\mathbf{k}'\lambda'} e^{2\pi i \mathbf{k}'\cdot\mathbf{r}} \frac{\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}'\lambda'}}{k'} \sin(2\pi k'ct)$$

$$\times \int d\mathbf{k}\, \boldsymbol{\varepsilon}_{-\mathbf{k}\lambda} e^{2\pi i \mathbf{k}\cdot\mathbf{r}} \frac{\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{-\mathbf{k}\lambda}}{k} \sin(2\pi kct)$$

$$= \sum_{\lambda=1,2} \sum_{\lambda'=1,2} \frac{4e^2}{c^2} \int d\mathbf{k}' \frac{\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}'\lambda'}}{k'} \sin(2\pi k'ct) \int d\mathbf{k} \frac{\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{-\mathbf{k}\lambda}}{k}$$

$$\times \sin(2\pi kct)\, \delta_{\lambda\lambda'}\, \delta(\mathbf{k}+\mathbf{k}')$$

$$= \sum_{\lambda=1,2} \frac{4e^2}{c^2} \int d\mathbf{k} \frac{(\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\lambda})^2}{k^2} \sin^2(2\pi kct)$$

$$\int \mathbf{H}^2\, d\mathbf{r} = \sum_{\lambda=1,2} \sum_{\lambda'=1,2} \frac{-4e^2}{c^2} \int d\mathbf{r} \int d\mathbf{k}'[\mathbf{k}'\times\boldsymbol{\varepsilon}_{\mathbf{k}'\lambda'}]\, e^{2\pi i \mathbf{k}'\cdot\mathbf{r}} \frac{\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}'\lambda'}}{k'^2}$$

$$\times \cos(2\pi k'ct)$$

$$\times \int d\mathbf{k}[\mathbf{k}\times\boldsymbol{\varepsilon}_{-\mathbf{k}\lambda}]\, e^{2\pi i \mathbf{k}\cdot\mathbf{r}} \frac{\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{-\mathbf{k}\lambda}}{k^2} \cos(2\pi kct)$$

$$= \sum_{\lambda=1,2} \int \frac{4e^2}{c^2} \int d\mathbf{k} \frac{\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\lambda}}{k^2} \cos^2(2\pi kct)$$

Hence the energy radiated is

$$R = \frac{1}{8\pi}\int (E^2 + H^2)\, d\mathbf{r}$$

$$= \frac{e^2}{2\pi c^2} \sum_{\lambda=1,2} \int d\mathbf{k} \frac{(\Delta\mathbf{v}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}\lambda})^2}{k^2}$$

$$= \frac{e^2}{2\pi c^2}|\Delta\mathbf{v}|^2 \int_0^{2\pi} d\varphi \int \sin\theta\, d\theta \sin^2\theta \int \frac{d\nu}{c}$$

where polar coordinates with the polar axis along $\Delta\mathbf{v}$ have been used. This gives

$$R = \frac{4e^2}{3c^3}(\Delta\mathbf{v})^2 \int d\nu$$

The integral over ν appears to give an infinite result, but our method is only valid for $\nu \ll 1/\tau$, where τ is the collision time. Thus our method is not valid to give the total energy radiated, but we can get the energy radiated at the frequency ν in the range where ν satisfies the above condition. The result

$$R_\nu = \frac{4e^2}{3c^3}(\Delta\mathbf{v})^2$$

agrees with the value obtained from the first method.

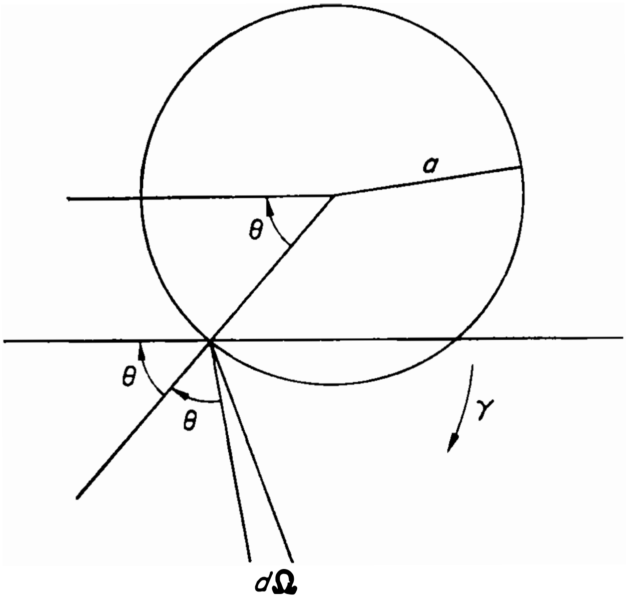
If we consider the scatterer to be an impenetrable sphere of radius a, the collision time is an instant, and so our calculation holds for all ν. From quantum theoretical arguments, $R_\nu = 0$ for $\nu > E_0/h$ where E_0 is the initial energy of the electron. Hence

$$R_\nu d\nu \begin{cases} = \dfrac{4e^2}{3c^3}|\Delta\mathbf{v}|^2 d\nu & \text{for} \quad 0 < \nu < E_0/h \\ = 0 \quad \text{for} \quad \nu > E_0/h \end{cases}$$

Now the cross-section $d\sigma$ for scattering in solid angle $d\boldsymbol{\Omega}$ is, for a solid sphere,

$$d\sigma = \frac{a^2}{4}\, d\boldsymbol{\Omega}$$

By the term cross-section, we mean that if N electrons are incident on the sphere per unit area (normal to the direction of the beam) per unit time, then the number scattered per unit time into the solid angle $d\boldsymbol{\Omega}$ is $N\,d\sigma$.



Thus the number of particles arriving at an element of area $a^2 \sin\theta\, d\theta\, d\varphi\, d$ is

$$Na^2 \sin\theta \cos\theta\, d\theta\, d\varphi$$

and since we have the relation

$$\gamma = \pi - 2\theta$$

this number is equal to

$$\frac{N}{4} a^2 \sin\gamma\, d\gamma\, d\varphi$$

These particles are scattered into the solid angle

$$d\boldsymbol{\Omega} = \sin\gamma\, d\gamma\, d\varphi$$

and hence we have derived the relation

$$d\sigma = \frac{a^2}{4} d\boldsymbol{\Omega}$$

and

$$\sigma = \pi a^2$$

Now

$$(\Delta \mathbf{v})^2 = v^2 + v'^2 - 2vv' \cos\gamma$$

which for elastic scattering reduces to

$$(\Delta \mathbf{v})^2 = 2v^2(1 - \cos\gamma)$$

Let us introduce a quantity $d\omega\, d\nu$ which is the cross-section $d\sigma$ multiplied by the energy radiated in frequency ν; that is

$$d\omega\,(\mathbf{\Omega}, \nu)\, d\nu = \frac{a^2}{4}\, d\mathbf{\Omega}\, \frac{4e^2}{3c^3}\, 2v^2(1 - \cos\gamma)\, d\nu$$

Integrating over $\mathbf{\Omega}$ we get

$$\omega(\nu)\, d\nu = \frac{8\pi e^2 a^2 v^2}{3c^3}\, d\nu$$

Let

$$\begin{aligned} \text{Э} &= \frac{\int_0^{E_0/h} \omega(\nu)\, d\nu}{E_0} \\ &= \frac{8\pi e^2 a^2 v^2}{3c^3 h} \\ &= \pi a^2 \times \frac{8e^2}{3hc}\, \frac{v^2}{c^2} \end{aligned}$$

Э gives a measure of the energy loss times its probability and is called the cross-section for energy loss. The mean energy loss ΔE in time t is

$$\Delta E = -Nt\text{Э}E$$

in our case

$$\text{Э} \approx \sigma \times \frac{1}{300} \times \beta^2$$

From this we can get a rough estimate of the efficiency of an X-ray tube. Treating atoms as hard spheres is a very crude approximation, and better results may be obtained by using a more refined model.

15 RADIATION REACTION

Let us now consider some effects of the radiation on the charge radiating it. We shall restrict ourselves to the case $v \ll c$ since we can always make a Lorentz transformation to a system where this condition holds. From (13.2) the energy radiated per unit time from a charge with acceleration $\dot{\mathbf{v}}$ is

$$\frac{2e^2\dot{\mathbf{v}}^2}{3c^3}$$

The possible sources of this energy are 1) energy in the field, i.e. $\frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{H}^2)$ and 2) energy of the charge, kinetic or potential. Now if the charge executes periodic motion, the field energy may vary but in a periodic way so that the energy of radiation must come from that of the charge. We shall find that there is a force of radiation reaction, **F** which causes damping and hence decreases the energy of the charge. Thus, strictly speaking, there is no real harmonic oscillator in nature. The reaction does work at the rate $\mathbf{F}\cdot\mathbf{v}$ so that in order to conserve energy we would like to have

$$\mathbf{F}\cdot\mathbf{v} + \frac{2}{3}\frac{e^2\dot{\mathbf{v}}^2}{c^3}$$

zero. However, this condition is too stringent, and we actually have it equal to

$$\frac{d}{dt}f(\mathbf{v}, \dot{\mathbf{v}})$$

so that the energy radiated over a certain interval of time is a constant which does not depend on the length of the interval. In fact, it takes into account the difference of the energies in the field at the initial and final times. One possible expression for **F** is

$$\mathbf{F} = \frac{2}{3}\frac{e^2}{c^3}\ddot{\mathbf{v}} \tag{15.1}$$

for then

$$\ddot{\mathbf{v}}\cdot\mathbf{v} + \dot{\mathbf{v}}\cdot\dot{\mathbf{v}} = \frac{d}{dt}(\dot{\mathbf{v}}\cdot\mathbf{v})$$

and

$$\int_{t_1}^{t_2}\mathbf{F}\cdot\mathbf{v}\,dt + \int_{t_1}^{t_2}\frac{2e^2}{3c^3}\dot{\mathbf{v}}^2\,dt = \left[\frac{2e^2}{3c^3}(\dot{\mathbf{v}}\cdot\mathbf{v})\right]_{t_1}^{t_2} \tag{15.2}$$

with the physical meaning that change in the energy of the charge + energy radiated = difference of the energies in the field.

This ensures conservation of energy, but in the case $\dot{\mathbf{v}}$ = const., we have a paradox that energy is radiated though there is no radiation reaction. Where does the energy of the radiation come from?

The expression for **F** written above turns out to be the correct one as we shall see when give a rigorous derivation of the formula. Before giving this proof, let us make a few applications.

The equation of motion of an uncharged harmonic oscillator is

$$m\ddot{x} + m\omega^2 x = 0$$

and the solution such that at $t = 0$, $x = a$, and $\dot{x} = 0$ is

$$x = a\cos\omega t$$

Now suppose the oscillator is suddenly charged, then it will start radiating, and the equation of motion is now

$$\ddot{x} + \omega^2 x = \frac{2e^2\dddot{x}}{3mc^3}$$

Let us assume the effect of the new term is small and try to get a solution of the form

$$x = ae^{-\gamma t/2}\cos\omega t$$

This will be valid if $\gamma \ll \omega$. Putting this in the differential equation, we have

$$\frac{2\gamma\omega}{2}ae^{-\gamma t/2}\sin\omega t + \frac{\gamma^2}{4}ae^{-\gamma t/2}\cos\omega t$$

$$= \frac{2e^2}{3mc^3}\left\{\omega^3 ae^{-\gamma t/2}\sin\omega t + \frac{3\gamma\omega^2 a}{2}e^{-\gamma t/2}\cos\omega t - \frac{3\gamma^2\omega a}{4}e^{-\gamma t/2}\sin\omega t - \frac{\gamma^3 a}{8}e^{-\gamma t/2}\cos\omega t\right\}$$

The first approximation is

$$\gamma = \frac{2e^2\omega^2}{3mc^3} = \frac{8\pi^2e^2\nu^2}{3mc^3} \tag{15.3}$$

and the condition $\gamma \ll \omega$ means

$$\omega \ll \frac{3mc^3}{2e^2}$$

For electrons, this means

$$\omega \ll \frac{mc^2}{e^2} \times c = \frac{3 \times 10^{10}}{2.8 \times 10^{-13}} \sim 10^{23}\ \text{Hz}$$

Thus the condition holds for all radiation except for the very high energy photons in cosmic rays.

The energy of the oscillator is

$$\frac{m\dot{x}^2}{2} + \frac{m\omega^2 x^2}{2} = \frac{ma^2\omega^2}{2} e^{-\gamma t}$$

so that the rate of decrease of energy is γ and

$$\frac{\text{decrease in energy per cycle}}{\text{total energy}} = \frac{2\pi\gamma}{\omega} = \frac{8\pi^2 e^2 \nu}{3mc^3}$$

We saw in (13.4) that

$$\frac{\text{energy radiated per cycle}}{\text{total energy}} = \frac{8\pi^2 e^2 \nu}{3mc^3}$$

so that energy is conserved over a complete cycle.

Exercise 19 Calculate how much charge the earth must have in order that the radiation reaction shorten the length of the year by a day per century. Assume $v/c \ll 1$ and that the orbit of the earth is a circle.

Due to damping, the radiation from an oscillator is not strictly monochromatic. Let us investigate the spectral distribution of the radiation. We shall let ν denote the frequency of the oscillator and f the frequency of the radiation. From (14.2) we have

$$R_f = \int d\Omega \frac{cr^2}{2\pi} |\mathbf{E}_f|^2$$

and from (14.1)

$$E_f \propto \int_0^\infty dt\, \ddot{x} e^{-2\pi i f t}$$

if we start off the oscillator at $t = 0$. This yields

$$E_f \propto \nu^2 \int_0^\infty dt\, e^{-\gamma t/2} \cos 2\pi\nu t e^{-2\pi i f t}$$

$$= \frac{\nu^2}{2} \int_0^\infty dt \{e^{-\gamma t/2 + 2\pi i(\nu - f)t} + e^{-\gamma t/2 - 2\pi i(\nu + f)t}\}$$

$$= \frac{\nu^2}{2} \left\{ -\frac{1}{2\pi i(\nu - f) - \gamma/2} + \frac{1}{2\pi i(\nu + f) + \gamma/2} \right\}$$

Now ν, the frequency of the oscillator, is taken to be positive. Thus the first term will be in resonance for $f = \nu$ and the second for $f = -\nu$. This gives for $|f| \sim \nu$

$$R_f \propto \frac{1}{4\pi^2(\nu - f)^2 + \gamma^2/4}$$

$$R_f = \frac{\varkappa}{(\nu - f)^2 + \gamma/4\pi^2} \tag{15.4}$$

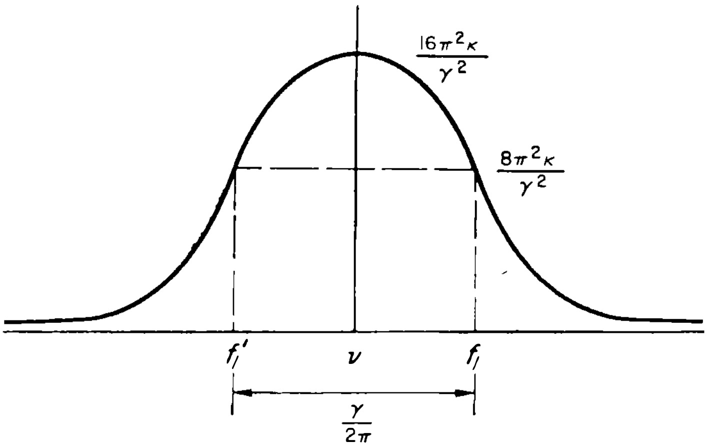
where f now takes on only positive values. For $f = \nu$, R_f has the maximum value

$$\frac{16\pi^2\varkappa}{\gamma^2}$$

Its value drops to half of this for $f = f_1$ such that

$$(\nu - f_1)^2 = \left(\frac{\gamma}{4\pi}\right)^2$$

and hence the half-width of the line is $\gamma/2\pi = \dfrac{4\pi e^2\nu^2}{3mc^3}$. This shows that the half-width of the line is proportional to ν^2.



The expression (15.4) for the spectral distribution of the energy is quite general. With a slight modification in the interpretation of the symbols according to the ideas of the quantum theory, it can be applied to the breadth of lines in atomic spectra. Let us assume the distribution to be of the form

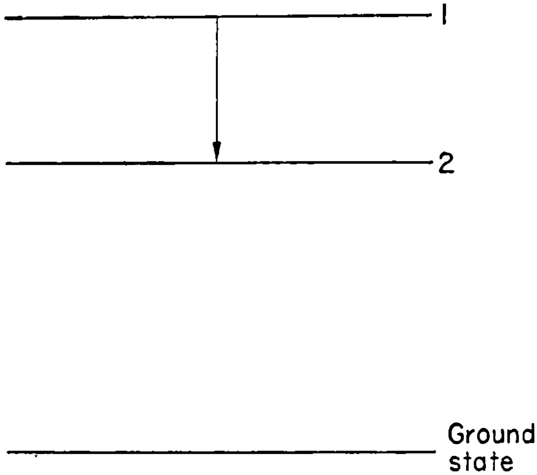
$$R_f = \frac{\varkappa}{(\nu - f)^2 + (\delta/4\pi)^2}$$

and try to find δ. Consider two excited states, 1 and 2. If λ_1, and λ_2 are the reciprocal mean lifetime of these states, so that

$$\dot{N}_1 = -\lambda_1 N_1$$

$$N_1 = N_1(0)\, e^{-\lambda_1 t}$$

$$N_2 = N_2(0)\, e^{-\lambda_2 t}$$



then by the uncertainty principle, the energies E_1 and E_2 of these two states will have widths given by

$$\Delta E_1 \gtrsim \hbar\lambda_1$$

$$\Delta E_2 \gtrsim \hbar\lambda_2$$

and the width of the line corresponding to the transition from state 1 to 2 will be

$$\Delta\nu \sim \frac{\Delta E_1}{h} + \frac{\Delta E_2}{h}$$

$$\gtrsim \frac{\lambda_1 + \lambda_2}{2\pi}$$

Thus

$$\frac{\delta}{2\pi} = \frac{\lambda_1 + \lambda_2}{2\pi}$$

and

$$R_f = \frac{\varkappa}{(\nu - f)^2 + \left(\dfrac{\lambda_1 + \lambda_2}{4\pi}\right)^2} \tag{15.5}$$

This formula was derived from very rough arguments, but it is correct. A very striking illustration of the formula occurs in some lines of stellar spectra. The center of a line may be absorbed out but the wings which may be as much as 20 Å apart are observed.

EXERCISE 20 Calculate the line breadth for the transition $2p$ — $1s$ in a hydrogen atom by (15.5) using the data given in Exercise 11. Also calculate the classical value for the frequency corresponding to that of the $2p$ — $1s$ transition, and compare the two values.

Next let us consider the force of radiation on a free particle. The equation of motion is

$$m\dot{v} = \frac{2e^2}{3c^3}\ddot{v}$$

or

$$\dddot{x} - \frac{3mc^3}{2e^2}\ddot{x} = 0$$

The solution consists of the usual part

$$x = a + vt$$

and another part. If we put

$$\ddot{x} = Ae^{\alpha t}$$

then

$$\dddot{x} = A\alpha e^{\alpha t}$$

and for

$$\alpha = \frac{3mc^3}{2e^2}$$

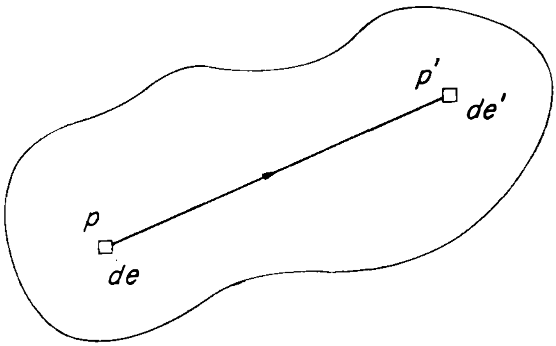
we have a solution,

$$\dot{x} = \frac{A}{\alpha}(e^{\alpha t} - 1) + v$$

$$x = \frac{A}{\alpha}\left\{\frac{1}{\alpha}(e^{\alpha t} - 1) - t\right\} + vt + a \tag{15.6}$$

Now for electrons, $1/\alpha \sim 10^{-23}$ sec., so this equation says that if $\ddot{x} = A \neq 0$ at $t = 0$ then an electron will instantly acquire a very large velocity and shoot off to infinity. Thus there is something drastically wrong. This shows that we must be careful in using the radiation reaction force. We shall derive the expression for the radiation reaction by considering the self-force and give the conditions of validity of the expression.

The calculation of the self-force on an electron was first given by Lorentz. We assume the charge distribution is rigid, so that at any instant each element of charge will have the same velocity. We also assume $v \ll c$ and shall keep only the terms linear in $\mathbf{v}$, $\dot{\mathbf{v}}$, $\ddot{\mathbf{v}}$ etc. Since the magnetic field is



proportional to v and hence the self-force due to it to v^2, we shall neglect this force, and consider just the self-force due to the electric field.

The potentials at p due to the element of charge de' is

$$d\varphi = \left[\frac{de'}{r(1 - v_{R/c})}\right]_{\text{ret}}$$

$$d\psi = \left[\frac{de'\mathbf{v}}{rc(1 - v_{R/c})}\right]_{\text{ret}}$$

We shall expand all retarded quantities in powers of $\tau = t - t' = r_{\text{ret}}/c$. Thus by Taylor's expansion

$$\mathbf{r}_{\text{ret}} = \mathbf{r} - \mathbf{v}\tau + \frac{\dot{\mathbf{v}}\tau^2}{2} - \frac{\ddot{\mathbf{v}}\tau^3}{6} + \cdots$$

$$\mathbf{v}_{\text{ret}} = \mathbf{v} - \dot{\mathbf{v}}\tau + \frac{\ddot{\mathbf{v}}\tau^2}{2} - \frac{\dddot{\mathbf{v}}\tau^3}{6} + \cdots$$

from which we get

$$r_{\text{ret}}^2 = r^2 - 2(\mathbf{r}\cdot\mathbf{v})\,\tau + (\mathbf{r}\cdot\dot{\mathbf{v}})\,\tau^2 - \frac{(\mathbf{r}\cdot\ddot{\mathbf{v}})\,\tau^3}{3} + \cdots$$

$$r_{\text{ret}} = r\left\{1 - \frac{(\mathbf{r}\cdot\mathbf{v})\,\tau}{r^2} + \frac{(\mathbf{r}\cdot\dot{\mathbf{v}})\,\tau^2}{2r^2} - \frac{(\mathbf{r}\cdot\ddot{\mathbf{v}})\,\tau^3}{6r^2}\right\} + \cdots$$

$$(\mathbf{r}\cdot\mathbf{v})_{\text{ret}} = -(rv_R)_{\text{ret}} = (\mathbf{r}\cdot\mathbf{v}) - (\mathbf{r}\cdot\dot{\mathbf{v}})\,\tau + \frac{(\mathbf{r}\cdot\ddot{\mathbf{v}})\,\tau^2}{2} - \frac{(\mathbf{r}\cdot\dddot{\mathbf{v}})\,\tau^3}{6} + \cdots$$

To our present approximation we therefore have

$$\tau = \frac{r_{\text{ret}}}{c} = \frac{r}{c} - \frac{(\mathbf{r}\cdot\mathbf{v})}{c} + \frac{(\mathbf{r}\cdot\dot{\mathbf{v}})\,r}{2c^2} - \frac{(\mathbf{r}\cdot\ddot{\mathbf{v}})\,r^2}{6c^3} + \cdots$$

$$r_{\text{ret}} = r - \frac{(\mathbf{r}\cdot\mathbf{v})}{c} + \frac{(\mathbf{r}\cdot\dot{\mathbf{v}})\,r}{2c^2} - \frac{(\mathbf{r}\cdot\ddot{\mathbf{v}})\,r^2}{6c^3} + \cdots$$

$$-\frac{(rv_R)_{\text{ret}}}{c} = \frac{(\mathbf{r}\cdot\mathbf{v})}{c} - \frac{(\mathbf{r}\cdot\dot{\mathbf{v}})\,r}{c^2} + \frac{(\mathbf{r}\cdot\ddot{\mathbf{v}})r^2}{2c^3} - \frac{(\mathbf{r}\cdot\dddot{\mathbf{v}})\,r^3}{6c^4} + \cdots$$

$$\left(r - \frac{rv_R}{c}\right)_{\text{ret}} = r - \frac{(\mathbf{r}\cdot\dot{\mathbf{v}})\,r}{2c^2} + \frac{(\mathbf{r}\cdot\ddot{\mathbf{v}})\,r^2}{3c^3} + \cdots$$

$$\left[\frac{1}{r(1 - v_R/c)}\right]_{\text{ret}} = \frac{1}{r}\left\{1 + \frac{(\mathbf{r}\cdot\dot{\mathbf{v}})}{2c^2} - \frac{(\mathbf{r}\cdot\ddot{\mathbf{v}})\,r}{3c^3} + \cdots\right\}$$

and hence

$$d\varphi = \frac{de'}{r}\left\{1 + \frac{(\mathbf{r}\cdot\dot{\mathbf{v}})}{2c^2} - \frac{(\mathbf{r}\cdot\ddot{\mathbf{v}})\,r}{3c^3} + \cdots\right\}$$

$$d\psi = \frac{de'}{cr}\left\{\mathbf{v} - \dot{\mathbf{v}}\tau + \frac{1}{2}\ddot{\mathbf{v}}\tau^2 - \cdots\right\}\left\{1 + \frac{(\mathbf{r}\cdot\dot{\mathbf{v}})}{2c^2} - \cdots\right\}$$

Since we are keeping only terms linear in $\mathbf{v}$ and its derivatives, the final factor in the hast expression may be dropped, yielding

$$d\psi = \frac{de'}{cr}\left\{\mathbf{v} - \dot{\mathbf{v}}\frac{r}{c} + \frac{\ddot{\mathbf{v}}r^2}{2c^2} - \cdots\right\}$$

From

$$d\mathbf{E} = +\nabla\, d\varphi - \frac{1}{c}\,\frac{\partial}{\partial t}\,d\psi,$$

the $+$ sign being used with the ∇ symbol because the vector $\mathbf{r}$ points from p to p' rather than the other way around, we now get

$$\begin{aligned} d\mathbf{E} &= -de'\frac{\mathbf{r}}{r^3}\left\{1 + \frac{(\mathbf{r}\cdot\dot{\mathbf{v}})}{2c^2} - \frac{(\mathbf{r}\cdot\ddot{\mathbf{v}})r}{3c^3} + \cdots\right\} \\ &\quad + de'\frac{1}{r}\left\{\frac{\dot{\mathbf{v}}}{2c^2} - \frac{\ddot{\mathbf{v}}r}{3c^3} - \frac{(\mathbf{r}\cdot\ddot{\mathbf{v}})\,\mathbf{r}}{3c^3 r} + \cdots\right\} \\ &\quad - de'\frac{1}{c^2 r}\left\{\dot{\mathbf{v}} - \ddot{\mathbf{v}}\frac{r}{c} + \dddot{\mathbf{v}}\frac{r^2}{2c^2} - \cdots\right\} \\ &= d\mathbf{E}^1 + d\mathbf{E}^2 + d\mathbf{E}^3 + d\mathbf{E}^4 \end{aligned}$$

where

$$d\mathbf{E}^1 = -de'\frac{\mathbf{r}}{r^3}$$

$$d\mathbf{E}^2 = -de'\frac{1}{2c^2r}\left(\mathbf{1} + \frac{\mathbf{rr}}{r^2}\right)\cdot\dot{\mathbf{v}}$$

$$d\mathbf{E}^3 = de'\frac{2}{3c^3}\ddot{\mathbf{v}}$$

$$d\mathbf{E}^4 = de'\cdot 0\left(\frac{\dddot{\mathbf{v}}r}{c^4}\right)$$

($0(z)$ means of the order of z.) The force on the element of charge de at P is

$$d\mathbf{F} = de\, d\mathbf{E}$$

and the total self-force is the double integral over $de\, de'$. The different parts of $\mathbf{E}$ give the following forces

$$\mathbf{F}^1 = +\int\int de\, d\mathbf{E}^1$$

$$= -\int\int d\mathbf{r}\, d\mathbf{r}'\, \varrho\varrho'\frac{\mathbf{r}}{r^3} \tag{15.7}$$

This is the force which tends to blow up the electron, but it gives no net force since reversing the sign of $\mathbf{r}$ just changes the sign of the integrand

$$\mathbf{F}^2 = \int\int de\, d\mathbf{E}^2$$

$$= -\frac{1}{2c^2}\int\int d\mathbf{r}\, d\mathbf{r}'\, \varrho\varrho'\frac{1}{r}\left(\mathbf{1} + \frac{\mathbf{rr}}{r^2}\right)\cdot\dot{\mathbf{v}}$$

For a spherically symmetric distribution, the tensor $\mathbf{rr}$ may be replaced by $\frac{1}{3}r^2\cdot\mathbf{1}$. Therefore

$$\mathbf{F}^2 = -\frac{2}{3}\frac{\dot{\mathbf{v}}}{c^2}\int\int d\mathbf{r}\, d\mathbf{r}'\frac{\varrho\varrho'}{r}$$

$$= -\frac{4}{3}\frac{\dot{\mathbf{v}}}{c^2}U \tag{15.8}$$

where U is the electrostatic self energy of the electron:

$$U = \frac{1}{2} \int\int dr\, dr' \frac{\varrho\varrho'}{r} = \frac{e^2}{a}$$

where a is the classical radius of the electron. We shall return to the discussion of U later.

$$\begin{aligned} \mathbf{F}^3 &= \int\int de\, d\mathbf{E}^3 \\ &= \frac{2\ddot{\mathbf{v}}}{3c^3} \int\int d\mathbf{r}\, d\mathbf{r}'\, \varrho\varrho' \\ &= \frac{2\ddot{\mathbf{v}}}{3c^3} e^2 \end{aligned} \tag{15.9}$$

This is the radiation reaction force, and, as it is the same as (15.1) we have justified it.

$$\mathbf{F}^4 = 0\left(\frac{\dddot{\mathbf{v}}}{c^4} ae^2\right)$$

and for an electron

$$\frac{|\dddot{\mathbf{v}}|\, ae^2}{c^4} \sim |\dddot{\mathbf{v}}| \times 10^{-63} \text{ dynes}$$

so that in general this force is negligible. It is important to note that these higher terms depend on the structure of the electron.

The force of radiation reaction acts not only on itself but on any charge nearby if $\frac{|\dot{\mathbf{v}}|}{|\mathbf{v}|} \frac{d}{c} \ll 1$. This condition is roughly equal to $d \ll \lambda$ since for a harmonic oscillator, $\frac{|\dot{\mathbf{v}}|}{|\mathbf{v}|} = \omega$. This is to be expected since if two charges near together oscillate with amplitudes A and A', the intensity of the radiation varies as $(A + A')^2$ and not as $A^2 + A'^2$. Hence each oscillator must damp the other in order to conserve energy.

EXERCISE 21 Calculate the force between two oscillators near together as a function of their phases and amplitudes.

16 SELF-ENERGY OF THE ELECTRON

We see from the expression (15.8) for $\mathbf{F}^2$ that a charged body has more inertia than an uncharged body. The total inertial force of a body is

$$\mathbf{F}_{in} = -m\dot{\mathbf{v}}$$

where m is its experimental mass. The electromagnetic contribution to this force is

$$\mathbf{F}_{em} = -\frac{4e^2}{3c^2a}\dot{\mathbf{v}}$$

We have

$$\frac{F_{em}}{F_{in}} \approx \frac{e^2}{mc^2a}$$

For macroscopic problems $F_{\mathrm{em}} \ll F_{\mathrm{in}}$ and F_{em} may be neglected. However, for the elementary particle, we cannot separate the two. Now F_{em} cannot be greater than F_{in}, and it seems a plausible assumption that the two are of the same order of magnitude. This means

$$a \sim \frac{e^2}{mc^2}$$

$$= 2.8 \times 10^{-13} \text{ cm}$$

for the electron. If the electron had only electromagnetic mass, the equation of motion could be written

$$\mathbf{F} = \mathbf{F}_s + \mathbf{F}_e = 0$$

where $\mathbf{F}_s$ is the self force and $\mathbf{F}_e$ is the external force acting on the electron:

$$\mathbf{F}_e = e\left\{\mathbf{E}_{\mathrm{ext}} + \left[\frac{\mathbf{v}}{c} \times \mathbf{H}_{\mathrm{ext}}\right]\right\}$$

In this case the whole idea of mass is unnecessary. This was Abraham's idea, but the theory does not work. There must be some non-electromagnetic force to keep the electron from blowing up. The fact that a static distribution of charge cannot be in equilibrium without any external force is Earnshaw's theorem. The theorem states that a charged body cannot rest in stable equilibrium under the influence of electric fields alone. It might be thought that a suitable distribution of current and charge in an electron would keep it together, but a quite general proof can be given to show that this is not possible. Consider

$$\mathbf{f} = \varrho\mathbf{E} + \mathbf{j} \times \mathbf{H}$$

The equation of motion is

$$\mathbf{f} = 0$$

if $\mathbf{E}$ and $\mathbf{H}$ include both external and self fields. A solution of $\mathbf{f} = 0$ for $\mathbf{E}$ and $\mathbf{H}$ just external fields is possible since one solution is $\mathbf{E} = 0$ and $\mathbf{H}$

parallel to **j**. However, there is no solution if **E** and **H** include the self fields. For we can write

$$\mathbf{j} \times \mathbf{H} = -\varrho \mathbf{E}$$

and the right side vanishes only if $\varrho = 0$ since we have $\nabla \cdot \mathbf{E} = \varrho$. Thus we have three non-homogeneous linear equations for j_x, j_y, and j_z with the determinant of their coefficients

$$\begin{vmatrix} 0 & H_z & -H_y \\ -H_z & 0 & H_x \\ H_y & -H_x & 0 \end{vmatrix} = 0$$

since it is a general theorem that the determinant of an antisymmetric matrix of odd rank is zero. Hence there is no non-zero solution for **j**.

Thus a part m_0 of the mass must be of non-electromagnetic origin, and the equation of motion is

$$m_0 \dot{\mathbf{v}} = \mathbf{F}_s + \mathbf{F}_e$$

If the calculation of chapter 15 is valid, we have

$$\mathbf{F}_s = -\frac{4U}{3c^2}\dot{\mathbf{v}} + \frac{2e^2}{3c^3}\ddot{\mathbf{v}} + 0\left(\frac{e^2\dddot{\mathbf{v}}a}{c^4}\right)$$

and if $\dfrac{a\omega}{c} \ll 1$, we have

$$\left(m_0 + \frac{4U}{3c^2}\right)\dot{\mathbf{v}} = \mathbf{F}_e + \frac{2e^2}{3c^3}\ddot{\mathbf{v}}$$

For macroscopic bodies,

$$\frac{U}{c^2 M} \sim \frac{Q^2}{c^2 M d}$$

where Q is the charge, M the mass, and d the dimension of the body. For a pith ball with $Q = 5$ e.s.u. and $d = 1$ cm this becomes

$$\begin{aligned} \frac{U}{c^2 M} &\sim \frac{(Q/e)^2}{(M/m)d}\,\frac{e^2}{mc^2} \\ &\sim 3 \times 10^{-13}\,\text{cm}\,\frac{(Q/e)^2}{(M/m)\,d} \\ &\sim 3 \times 10^{-20} \end{aligned}$$

For the electron, however, U is not negligible.

In view of the instability of charges under the action of electromagnetic forces alone, we are led to two possibilities: 1) Maxwell's equations do not hold in regions comparable to the size of the electron. Born has developed a theory of this form, but it is not unique and there is as yet no evidence that it describes reality. An acceptable theory would show that

$$\frac{e^2}{\hbar c} = \frac{1}{137}$$

which contains both $\hbar$ and c. Thus the theory must be both quantum mechanical and relativistic. There is at present no satisfactory theory. 2) Accept the stability of the electron as a fact. The theory of relativity shows that $E = mc^2$ so that the factor 4/3 multiplying U is a direct indication of the fact that other forces are present. This force must necessarily be attractive and hence give rise to negative mass. It is reasonable to assume that it is equal to $-\frac{1}{3}U/c^2$ thus giving for the total mass just U/c^2.

Since we do not know the distribution of the charge inside the electron, the terms of higher order than the radiation reaction in the calculation of chapter 15 cannot be calculated. Hence we can only treat the problem in which $\dfrac{a\omega}{c} \ll 1$ where $a > 3 \times 10^{-13}$ cm, for the electron.

The solution of the paradox given in section 15 goes as follows: We obtained the solution of

$$m\ddot{x} - \frac{2e^2}{3c^3}\dddot{x} = 0$$

as

$$\dot{x} = v + \frac{A}{\alpha}(e^{\alpha t} - 1)$$

where

$$\alpha = \frac{3mc^3}{2e^2}$$

Thus $\omega \sim \dfrac{mc^3}{e^2}$ and $\dfrac{a\omega}{c} \sim \dfrac{mc^2a}{e^2}$. If $\dfrac{a\omega}{c}$ is not small, the neglect of the higher order terms is not justified, and the solution is not good. If $\dfrac{a\omega}{c} \ll 1$, then $\dfrac{mc^2a}{e^2} \ll 1$, or $\dfrac{U}{c^2} \gg m$. This means that in order to keep the total mass of the electron equal to m, there must be a large amount of negative non-electromagnetic mass. It is by increasing this negative mass that the kinetic energy of the electron can increase so explosively. Since we do not actually observe an electron shooting off to infinity by itself, it means that $U \sim mc^2$.

Whether the electron theory is applicable to a particular problem requires close analysis. Consider the Bremsstrahlung of very high energy electrons with emission of hard γ-rays which occurs in cosmic ray showers. It seems at first that on account of the velocity of the electrons being very close to c, and their very high energy, the problem cannot be treated. However, on using the coordinates moving with the electron, the γ-rays appear very soft on account of the Doppler effect, and the problem can be treated. The calculation has actually been verified by the observations on cascade showers.

17 CLASSICAL THEORY OF SCATTERING AND DISPERSION

In this chapter we shall treat a few problems in the classical electron theory of matter. For atomic electrons, $v/c \sim 1/100$, and, since optical light waves do not give the electron much additional velocity, it is justified to neglect relativistic effects. We shall consider matter to be composed of isotropic oscillators. Then an external field induces a moment m in such a body

$$\mathbf{m} = \alpha \mathbf{E}$$

For non-isotropic oscillators the expression is

$$\mathbf{m} = \alpha \cdot \mathbf{E}$$

where α is a tensor. For light waves, $\mathbf{E}$ is periodic, so the charges are accelerated and hence radiate. This accounts for the phenomena of scattering and dispersion. The two phenomena are related but are observed by two independent experiments.

Let us first suppose that the oscillators are harmonic: then considering just one dimension we have

$$m\ddot{x} + m\omega^2 x = eE_x$$

with a correction for the radiation reaction. We shall write

$$E_x = \mathscr{R}\{E_0 e^{i\nu t}\}$$

where $\mathscr{R}$ means the real part of. If a is the dimension of the oscillator, from the uncertainty principle we have

$$a \sim \frac{\hbar}{\sqrt{\mathbf{p}^2}}$$

and since

$$h\nu \sim E_{\text{osc}}$$

$$\lambda \sim \frac{c}{\nu} \sim \frac{c\,h}{E_{\text{osc}}}$$

we have

$$\frac{a}{\lambda} \sim \frac{E_{\text{osc}}}{c\sqrt{\mathbf{p}^2}} \sim \frac{v_{\text{osc}}}{c}$$

We shall write

$$\ddot{x} + \omega^2 x = \frac{e}{m} E_0 e^{i\nu t}$$

and take the real part of x in the solution. The solution is

$$x = Ae^{i\nu t} + Be^{i\omega t} + Ce^{-i\omega t}$$

where

$$(-\nu^2 + \omega^2)\,A = \frac{eE_0}{m}$$

$$A = \frac{eE_0}{m(\omega^2 - \nu^2)}$$

If we include the radiation reaction, we have

$$-\frac{1}{\nu_0}\dddot{x}\,x + \ddot{x} + \omega^2 x = \frac{e}{m} E_0 e^{i\nu t}$$

where

$$\frac{1}{\nu_0} = \frac{2e^2}{3mc^3}$$

and this gives

$$A = \frac{eE_0}{m\left(\omega^2 - \nu^2 + \dfrac{i\nu^3}{\nu_0}\right)}$$

We see that the phase relation changes continuously as ν passes through ω. For light waves $\nu_0 \gg \nu$, so the correction is very small, and its effect is only felt near resonance.

We see that the scattered light will consist only of frequencies ν and ω, and this is a misleading result arising from the special model of oscillators we chose. If they are not strictly harmonic, there would be frequencies of $\nu + \tau\omega$ where $\tau = 0, \pm 1, \pm 2, \ldots$ This is the Raman effect and is not a quantum mechanical effect. It arises from the fact that a general periodic motion with period T is given by

$$x = \mathscr{R}\left\{\sum_\tau A_\tau e^{i\omega\tau t}\right\}$$

where $\tau = 0, \pm 1, \pm 2, \ldots$ and $\omega T = 2\pi$. In general A_τ will be large only for low harmonics. A harmonic motion is a very special case where $A_\tau = 0$ for all τ except ± 1. In dispersion, we are only interested in the coherent radiation, and the model of the harmonic oscillator is good.

We shall first consider scattered waves of the same frequency as the incident wave. The induced moment is

$$e\mathbf{r} = \frac{(e^2/m)\mathbf{E}_0 e^{i\nu t}}{\omega^2 - \nu^2 + i\dfrac{\nu^3}{\nu_0}}$$

and therefore

$$e\ddot{\mathbf{r}} = -\frac{(e^2/m)\mathbf{E}_\nu e^{i\nu t}}{\dfrac{\omega^2}{\nu^2} - 1 + i\dfrac{\nu^3}{\nu_0}}$$

Remembering that the actual value is the real part, we have

$$e\ddot{\mathbf{r}} = -\frac{e^2}{m}\mathbf{E}_0 \frac{\left\{\left(\dfrac{\omega^2}{\nu^2} - 1\right)\cos \nu t + \dfrac{\nu}{\nu_0}\sin \nu t\right\}}{\left(\dfrac{\omega^2}{\nu^2} - 1\right)^2 + \dfrac{\nu^2}{\nu_0^2}} \tag{17.1}$$

Thus the rate of radiation is from (13.2)

$$\frac{2e^2}{3c^3}\ddot{r}^2 = \frac{2e^4E_0^2}{3m^2c^3}$$

$$\times \frac{\left\{\left(\dfrac{\omega^2}{\nu^2} - 1\right)^2 \cos^2 \nu t + 2\dfrac{\nu}{\nu_0}\left(\dfrac{\omega^2}{\nu^2} - 1\right)\sin \nu t \cos \nu t + \left(\dfrac{\nu}{\nu_0}\right)^2 \sin^2 \nu t\right\}}{\left\{\left(\dfrac{\omega^2}{\nu^2} - 1\right)^2 + \dfrac{\nu^2}{\nu_0^2}\right\}^2}$$

and the average rate is

$$\frac{e^4 E_0^2}{3m^2c^3} \frac{1}{\left\{\left(\dfrac{\omega^2}{\nu^2} - 1\right)^2 + \dfrac{\nu^2}{\nu_0^2}\right\}}$$

Now the flux of incident energy is

$$S = \frac{c}{4\pi} \mathbf{E}^2$$

and the average flux is

$$\frac{c}{8\pi} E_0^2$$

Thus the total corss section for scattering with frequency ν, which is the radiation scattered per unit flux, is

$$\sigma = \frac{8\pi e^4}{3m^2c^4} \frac{1}{\left\{\left(\dfrac{\omega^2}{\nu^2} - 1\right)^2 + \dfrac{\nu^2}{\nu_0^2}\right\}} \tag{17.2}$$

Since (13.2) is for dipole radiation, the angular distribution is $\sin^2 \theta$, where θ is the angle between $\mathbf{E}$ and the direction of observation. The limit $\omega = 0$ corresponds to scattering by free electrons, and in this case we have

$$\sigma \approx \frac{8\pi e^4}{3m^2c^4}$$

since we may neglect ν^2/ν_0^2 as there can be no resonance. This is the Thomson formula. Near resonance we have

$$\sigma \approx \frac{8\pi e^4}{3m^2c^4} \frac{1}{\left\{2^2\left(\dfrac{\omega}{\nu} - 1\right)^2 + \dfrac{\omega^2}{\nu_0^2}\right\}}$$

$$= \frac{2\pi e^4}{3m^2c^4} \frac{\nu^2}{\left\{(\omega - \nu)^2 + \left(\dfrac{\omega^2}{2\nu_0}\right)^2\right\}}$$

and the half-width of the line is

$$\frac{\omega^2}{\nu_0} = \frac{2e^2\omega^2}{3mc^3}$$

As stated earlier, the harmonic oscillator is a very specialized model, so we shall derive the induced moment for the general case. In terms of the angle and action variables β and J, the coordinate x of the oscillator is given by

$$x(t) = \frac{1}{2} \sum_{\tau} x_\tau(J)\, e^{2\pi i \tau\{\omega(J)t + \beta\}}$$

where $\tau = 0, \pm 1, \pm 2, \ldots$

The radiation is considered as a perturbation

$$V = exE_0 e^{2\pi i \nu t} = \varepsilon U$$

where $\varepsilon = eE_0$ si small. It must be remembered that we are to take the real part of the quantities involved. In our notation, ω is the natural fundamental frequency of the electron, and ν is the frequency of incident light. We make a contact transformation from the variables J, β to $\bar{J}, \bar{\beta}$ by means of a generating function $S(\bar{J}, \beta)$ and we do this by expanding S in powers of ε and, for the first order calculation, keeping only the terms linear in ε. Thus

$$S = S_0 + \varepsilon S_1 + \cdots$$

Since $S = S_0$ for $\varepsilon = 0$, S_0 must be the generating function for the identity transformation, namely

$$S_0 = \bar{J}\beta$$

Hence

$$J = \frac{\partial S}{\partial \beta} = \bar{J} + \varepsilon \frac{\partial S_1}{\partial \beta}$$

$$\bar{\beta} = \frac{\partial S}{\partial \bar{J}} = \beta + \varepsilon \frac{\partial S_1}{\partial \bar{J}}$$

Now

$$x(J, \beta) = x(\bar{J}, \bar{\beta}) + \frac{\partial x}{\partial \bar{J}}(J - \bar{J}) + \frac{\partial x}{\partial \beta}(\beta - \bar{\beta})$$

and therefore

$$\begin{aligned}
\delta x &\equiv x(\bar{J}, \bar{\beta}) - x(J, \beta) \\
&= \frac{\partial x}{\partial J}(\bar{J} - J) + \frac{\partial x}{\partial \beta}(\bar{\beta} - \beta) \\
&= \varepsilon \left\{ \frac{\partial x}{\partial \beta} \frac{\partial S_1}{\partial \bar{J}} - \frac{\partial x}{\partial J} \frac{\partial S_1}{\partial \beta} \right\} \\
&= \varepsilon(x, S_1)
\end{aligned}$$

where (x, S_1) is the Poisson bracket. To determine S_1, we have the Hamilton–Jacobi partial differential equation

$$\frac{\partial S}{\partial t} + H_0\left(\frac{\partial S}{\partial \beta}\right) + \varepsilon U = 0$$

Expanding to first order in ε, we get

$$\frac{\partial S_0}{\partial t} + \varepsilon \frac{\partial S_1}{\partial t} + H_0(\bar{J}) + \varepsilon \frac{\partial H_0}{\partial \bar{J}} \frac{\partial S_1}{\partial \beta} + \varepsilon U = 0$$

and since $\partial S_0/\partial t + H_0(\bar{J}) = 0$

$$\frac{\partial S_1}{\partial t} + \frac{\partial H_0}{\partial \bar{J}} \frac{\partial S_1}{\partial \beta} + U = 0$$

With the explicit form for U inserted this gives

$$\frac{\partial S_1}{\partial t} + \omega \frac{\partial S_1}{\partial \beta} + \frac{1}{2} \sum_\tau x_\tau e^{2\pi i(\tau w + \nu t)} = 0$$

where $w = \omega t + \beta$ is the usual angle variable and where we remember that $\omega(J) = \partial H_0/\partial J$.. To solve this equation, we make the "ansatz"

$$S_1 = \sum_\tau \sigma_\tau e^{2\pi i(\tau w + \nu t)}$$

Then the condition on σ_τ is

$$\sigma_\tau = - \frac{x_\tau}{4\pi i(\tau\omega + \nu)}$$

which yields

$$S_1 = - \frac{1}{4\pi i} \sum_\tau \frac{x_\tau}{(\tau\omega + \nu)} e^{2\pi i(\tau w + \nu t)}$$

$$\frac{\partial S_1}{\partial \beta} = - \frac{1}{2} \sum_\tau \frac{\tau x_\tau}{(\tau\omega + \nu)} e^{2\pi i(\tau w - \nu t)}$$

$$\frac{\partial S_1}{\partial J} = - \frac{1}{4\pi i} \sum_\tau \frac{\partial}{\partial J} \left\{\frac{x_\tau}{(\tau\omega + \nu)}\right\} e^{2\pi i(\tau w + \nu t)}$$

the dependence of ω on J being neglected in the last expression. We have also

$$\frac{\partial x}{\partial J} = \frac{1}{2} \sum_\tau \frac{\partial x_\tau}{\partial J} e^{2\pi i \tau w}$$

$$\frac{\partial x}{\partial \beta} = \pi i \sum_\tau \tau x_\tau e^{2\pi i \tau w}$$

Therefore

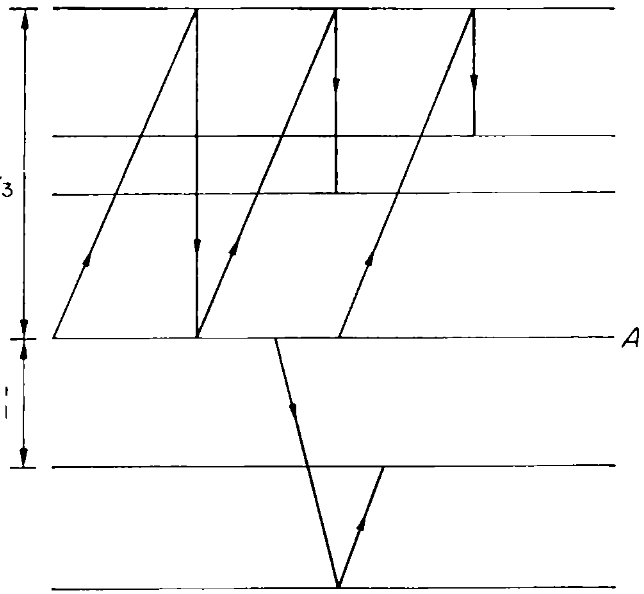
$$\delta x = -\frac{\varepsilon}{4}\sum_{\tau,\tau'}\left[\tau x_\tau \frac{\partial}{\partial J}\left\{\frac{x_{\tau'}}{\tau'\omega+\nu}\right\} e^{2\pi i\{(\tau+\tau')w+\nu t\}}\right.$$
$$\left. - \frac{\tau x_\tau}{(\tau\omega+\nu)}\frac{\partial x_{\tau'}}{\partial J} e^{2\pi i\{(\tau+\tau')w+\nu t\}}\right]$$

Remembering that we have to take the real part, the result is

$$e\delta x = -\frac{e^2E_0}{4}\mathscr{R}\sum_{\tau,\tau'}\tau x_\tau\left[\frac{\partial}{\partial J}\left\{\frac{x_{\tau'}}{\tau'\omega+\nu}\right\}\right.$$
$$\left. - \frac{1}{\tau\omega+\nu}\frac{\partial x_{\tau'}}{\partial J}\right] e^{2\pi i\{t[(\tau+\tau')\omega+\nu]+(\tau+\tau')\beta\}} \tag{17.3}$$

and hence the scattered radiation consists of frequencies $\nu + n\omega$, $n = 0$, ± 1, ± 2, ... and it is an accident that all terms with $\tau + \tau' \neq 0$ vanish in the case of the harmonic oscillator.

In applying the formula to atomic problems, quantum mechanical ideas must be used in the interpretation. An atom originally in state A may



undergo transitions some of which a reillustrated in the diagram. The frequencies of the scattered radiations are

$$\nu - \Delta\nu$$

where $\Delta\nu$ is the transition frequency from A to a higher level, and

$$\nu + \Delta\nu'$$

where $\Delta\nu'$ is the transition frequency from A to a lower level. This explains the Raman effect.

Let us now consider dispersion and absorption of radiation in a dielectric. For these considerations we have a good model for the dielectric if we assume it to consist of a large number of harmonic oscillators. We have seen that for a single oscillator, the induced moment is

$$\mathbf{m} = e\mathbf{x} = \frac{e^2\mathbf{E}_0}{m} \frac{e^{i\nu t}}{\omega^2 - \nu^2 + \dfrac{i\nu^3}{\nu_0}}$$

for an external field $\mathbf{E} = \mathbf{E}_0 e^{i\nu t}$. We need not consider the terms arising from the free vibration of the oscillator since by friction it is soon damped out. Let N_k be the number of oscillators of frequency ω_k in a unit volume. Then the induced moment per unit volume is

$$\mathbf{P} = \sum_k N_k e\mathbf{x}_k = \frac{e^2\mathbf{E}_0 e^{i\nu t}}{m} \sum_k \frac{N_k}{\omega_k^2 - \nu^2 + \dfrac{i\nu^3}{\nu_0}}$$

and hence the polarizability α is

$$\alpha = \sum_k \frac{(e^2/m)\, N_k}{\omega_k^2 - \nu^2 + \dfrac{i\nu^3}{\nu_0}} \tag{17.4}$$

For a static field, the polarizability α_0 is

$$\alpha_0 = \sum_k \frac{e^2 N_k}{m\omega_k^2}$$

We note that

$$[\alpha] = \frac{ML^3T^{-2}L^{-3}}{MT^{-2}} = 1$$

Thus α is a pure number.

EXERCISE 22 Derive the formula for the polarizability in a static field by considering the effect of a steady field on a collection of oscillators.

The propation of electromagnetic waves in a dielectric may be treated in either of two ways. The first method is to take Maxwell's equation in a material medium, and get

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{1}{\varepsilon}\Delta\mathbf{E} = 0 \tag{17.5}$$

Then use the relation between ε, the dielectric constant, and α

$$\frac{\varepsilon_\nu - 1}{\varepsilon_\nu + 2} = \frac{4\pi\alpha_\nu}{3}$$

which holds for any frequency ν. This equation is a consequence of the well-known relation between polarization and the external field,

$$\mathbf{P} = \alpha\left(\mathbf{E} + \frac{4}{3}\pi\mathbf{P}\right)$$

The second method which leads to the same result is to take Maxwell's equation for a vacuum and introduce the charge and current due to the oscillators. We have

$$\nabla \cdot \mathbf{E} = 4\pi\varrho$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + 4\pi\mathbf{j}$$

$$\nabla \times (\nabla \times \mathbf{E}) + \frac{1}{c}\frac{\partial}{\partial t}\nabla \times \mathbf{H} = 0$$

$$\nabla\nabla \cdot \mathbf{E} - \Delta\mathbf{E} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{4\pi}{c}\frac{\partial \mathbf{j}}{\partial t} = 0$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \Delta\mathbf{E} = -\frac{4\pi}{c}\frac{\partial \mathbf{j}}{\partial t} - 4\pi\nabla\varrho$$

We shall consider a definite frequency ν and write

$$\mathbf{E} = \mathbf{E}_{\max} e^{i\nu t} \tag{17.6}$$

Also, since most of the macroscopic properties of the body will depend only on the space averages of the quantities, we shall take the average over a volume containing a large numbers of oscillators, but with dimensions small compared to the wavelength λ. Then

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \rightarrow -\frac{\nu^2}{c^2}\bar{\mathbf{E}}$$

$$\Delta\mathbf{E} \rightarrow \overline{\Delta\mathbf{E}} \approx \Delta\bar{\mathbf{E}}$$

$$\mathbf{j} \rightarrow \bar{\mathbf{j}} = \frac{1}{c}\frac{\partial \mathbf{P}}{\partial t} = \sum_k N_k \frac{e}{c}\dot{\mathbf{x}}_k = \frac{i\nu}{c}\frac{e^2\bar{\mathbf{E}}_0}{m}\sum_k \frac{N_k}{\omega_k^2 - \nu^2 + \dfrac{i\nu^3}{\nu_0}}$$

$$\frac{4\pi}{c}\frac{\partial \mathbf{j}}{\partial t} \rightarrow \frac{4\pi i\nu}{c}\bar{\mathbf{j}} = -4\pi\frac{\nu^2}{c^2}\alpha_\nu\bar{\mathbf{E}}_0$$

$$\varrho \rightarrow \bar{\varrho} = -\nabla\cdot\mathbf{P} = -\alpha_\nu\,\nabla\cdot\bar{\mathbf{E}}_0$$

$$4\pi\,\nabla\,\varrho \rightarrow 4\pi\,\overline{\nabla\,\varrho} \approx 4\pi\,\nabla\bar{\varrho} = -4\pi\alpha_\nu\,\nabla\,\nabla\cdot\bar{\mathbf{E}}_0$$

The field $\bar{\mathbf{E}}_0$ is *not* the averaged field $\bar{\mathbf{E}}$. It is the average field which would exist at the location of an oscillator if that oscillator were absent, and it is given by

$$\bar{\mathbf{E}}_0 = \bar{\mathbf{E}} + \frac{4\pi}{3}\mathbf{P}$$

whence it follows that

$$\mathbf{P} = \alpha_\nu\left(\bar{\mathbf{E}} + \frac{4\pi}{3}\mathbf{P}\right) = \frac{\alpha_\nu}{1 - \dfrac{4\pi}{3}\alpha_\nu}\bar{\mathbf{E}}$$

The differential equation for $\mathbf{E}$ is now

$$-\frac{\nu^2}{c^2}\bar{\mathbf{E}} - \Delta\bar{\mathbf{E}} = \frac{4\pi\alpha_\nu}{1 - \dfrac{4\pi}{3}\alpha_\nu}\left(\frac{\nu^2}{c^2}\bar{\mathbf{E}} + \nabla\,\nabla\cdot\bar{\mathbf{E}}\right)$$

In the case of free wave propagation $\bar{\mathbf{E}}$ is transverse so that

$$\nabla\cdot\bar{\mathbf{E}} = 0$$

and the differential equation simplifies to

$$\Delta\bar{\mathbf{E}} + \frac{\nu^2}{c^2}\left\{1 + \frac{4\pi\alpha_\nu}{1 - \dfrac{4\pi}{3}\alpha_\nu}\right\}\bar{\mathbf{E}} = 0$$

The expression in curly brackets is equal to ε_ν, so our result agrees with that of the first method. The solution of this equation corresponding to the propagation of a plane wave in the direction of a unit vector $\mathbf{n}$ is

$$\bar{\mathbf{E}} = \mathbf{A}e^{i\nu t - i\varkappa\mathbf{n}\cdot\mathbf{r}}, \qquad \mathbf{A}\cdot\mathbf{n} = 0$$

where

$$\varkappa = \frac{\nu}{c}\sqrt{\frac{1 + \frac{8\pi}{3}\alpha_\nu}{1 - \frac{4\pi}{3}\alpha_\nu}}$$

For away from resonance α_ν is practically real, but near resonance it has an appreciable imaginary part which corresponds to absorption of the radiation. For a single resonant frequency we have

$$\varkappa^2 = \frac{\nu^2}{c^2}\frac{1 + \frac{8\pi}{3}\frac{e^2N}{m(\omega^2 - \nu^2 + i\nu^3/\nu_0)}}{1 - \frac{4\pi}{3}\frac{e^2N}{m(\omega^2 - \nu^2 + i\nu^3/\nu_0)}}$$

$$= \frac{\nu^2}{c^2}\frac{\omega^2 - \nu^2 + i\nu^3/\nu_0 + \frac{8\pi}{3}\frac{e^2N}{m}}{\omega^2 - \nu^2 + i\nu^3/\nu_0 - \frac{4\pi}{3}\frac{e^2N}{m}} \tag{17.7}$$

We see that the resonance frequency is not at ω but at $\bar{\omega}$ where $\bar{\omega}^2 = \omega^2 - \frac{4\pi}{3}\frac{e^2N}{m}$. If we write $\varkappa = \frac{\nu}{c}(n - is)$ then we have

$$n^2 - s^2 = 1 + \frac{4\pi e^2N}{m}\frac{\bar{\omega}^2 - \nu^2}{(\bar{\omega}^2 - \nu^2)^2 + (\nu^3/\nu_0)^2} \tag{17.8}$$

$$2ns = \frac{4\pi e^2N}{m}\frac{\nu^3}{\nu_0}\frac{1}{(\bar{\omega}^2 - \nu^2)^2 + (\nu^3/\nu_0)^2} \tag{17.9}$$

[n is the index of refraction of the dielectric, and $c/(\nu s)$ is the penetration depth. For $\nu > \bar{\omega}$ we enter a narrow region of anomalous dispersion where the index of refraction is less than unity. This does not, however, mean that a signal can be sent faster than light. Because of the strong absorption, a pulse of radiation will disappear before it moves a distance equal to its own width. Ed.]

EXERCISE 23 Calculate the value of the penetration depth at resonance.

Let us now interpret our result in the light of modern atomic theory. Actually, a dielectric is composed not of harmonic oscillators but of atoms in the normal state, and they have a set of excited states, and there is resonance for each possible transition with frequency given by

$$\nu = \omega_k = \frac{E_k - E_0}{\hbar}$$

The scattered radiation may not be of the same frequency as the absorbed frequency, and in this case we have the Raman effect, but this effect is small, and, since it is incoherent, it does not affect the propagation of the initial wave except that it gives a small absorption. In our model, this effect is neglected. Aside from this, the atoms act like a collection of oscillators. Each possible transition acts like an oscillator of frequency

$$\omega_k = \frac{E_k - E_0}{\hbar}$$

and the number of oscillators with frequency ω_k in a unit volume is

$$N_k = N_{\text{atoms}} f_k$$

where f_k is a proper fraction. In a single atom, each transition corresponds to a fraction f_k of an oscillator, and f_k is called the oscillator strength of the transition. If the values of f_k, ω_k are found either empirically or by quantum mechanical calculations and then put in (17.7) very good results are obtained. The quantum mechanical calculation for f_k and ω_k can only be done for a few simple cases, and there is no general argument by which we can find these values for complex atoms. For example, in one-valence-electron atoms, H, Na, Cs, the f_k corresponding to the transition from the normal state to the first excited state are 0.35, 0.975, 0.98 respectively.

18 HAMILTONIAN THEORY FOR THE MOTION OF A CHARGED PARTICLE IN AN ELECTROMAGNETIC FIELD

The equation of motion of a particle with mass m and charge e in an non-electromagnetic potential field V is

$$m\ddot{\mathbf{x}} = -\nabla V \tag{18.1}$$

If there is in addition an external electromagnetic field **E** and **H**, the equation is

$$m\ddot{\mathbf{x}} = -\nabla V + e\{\mathbf{E} + (\mathbf{v}/c) \times \mathbf{H}\} \tag{18.2}$$

We want to find a function H of the canonically conjugate variables p_i, x_i such that

$$\left.\begin{aligned} \dot{x}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial x_i} \end{aligned}\right\} \tag{18.3}$$

are equivalent to the equations of motion.

In the absence of an electromagnetic field, the function is

$$H_0(\mathbf{p}, \mathbf{x}) = \frac{1}{2m} p^2 + V(\mathbf{x})$$

since then Eqs. (18.3) become

$$\dot{x}_i = \frac{p_i}{m}$$

$$\dot{p}_i = -\frac{\partial V}{\partial x_i}$$

yielding

$$m\ddot{x}_i = -\frac{\partial V}{\partial x_i}$$

which is just (18.1).

When a field is present, we introduce the potentials defined by

$$\mathbf{E} = -\nabla \varphi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{H} = \nabla \times \mathbf{A}$$

and the function H for this case is

$$H = H_0\left(\mathbf{p} - \frac{e}{c}\mathbf{A}, \mathbf{x}\right) + e\varphi$$

For then Eqs. (18.3) become

$$\dot{x}_i = \frac{1}{m}\left(p_i - \frac{e}{c}A_i\right)$$

$$\dot{p}_i = -e\frac{\partial \varphi}{\partial x_i} - \frac{\partial V}{\partial x_i} - \sum_j \frac{1}{m}\left(p_j - \frac{e}{c}A_j\right)\left(-\frac{e}{c}\right)\frac{\partial A_j}{\partial x_i}$$

yielding

$$p_i = m\dot{x}_i + \frac{e}{c} A_i$$

$$m\ddot{x}_i = -\frac{e}{c}\frac{dA_i}{dt} - e\frac{\partial\varphi}{\partial x_i} - \frac{\partial V}{\partial x_i} + \frac{e}{c}\sum_{\delta} \dot{x}_j \frac{\partial A_j}{\partial x_i}$$

But since

$$\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + \sum_j \dot{x}_j \frac{\partial A_i}{\partial x_j}$$

we have

$$m\ddot{x}_i = -\frac{\partial V}{\partial x_i} - e\frac{\partial\varphi}{\partial x_i} - \frac{e}{c}\frac{\partial A_i}{\partial t} + \frac{e}{c}\sum_{j\neq i} \dot{x}_j \left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)$$

which is the same as (18.2).

This theory can be used to deduce Larmor's theorem very simply. The theorem states that the effect of a uniform magnetic field H on a charged particle describing a closed orbit is to leave the form of the orbit, its inclination to the magnetic lines of force, and the motion in the orbit unaltered, and merely leads to the addition of a uniform precession of the orbit about the direction of the lines of force, the precession velocity being

$$\omega = \frac{eH}{2mc}$$

Suppose the potential is radial so that $V = V(r)$. Then

$$H_0 = \frac{p^2}{2m} + V(r)$$

Let the external field be a uniform magnetic field H along the z-axis. Then we can take

$$A_x = -\frac{Hy}{2}$$

$$A_y = +\frac{Hx}{2}$$

$$A_z = 0$$

and

$$H = \frac{1}{2m}\left\{\left(p_x + \frac{eH}{2c}y\right)^2 + \left(p_y - \frac{eH}{2c}x\right)^2 + p_z^2\right\} + V(r)$$

This expression is the same as that obtained from H_0 by transforming to a coordinate system which rotates with angular velocity $\omega = eH/2mc$ about the z-axis, for then

$$\dot{x}' = \dot{x} + \frac{eH}{2mc} y$$

$$\dot{y}' = \dot{y} - \frac{eH}{2mc} x$$

$$\dot{z}' = \dot{z}$$

$$r' = r$$

This takes into account both the coriolis and centrifugal forces.

EXERCISE 24 By considering the effect of the Larmor precession on the propagation of the electromagnetic waves in a dielectric, give a theory of the Faraday effect.

CHAPTER 2

Special Theory of Relativity

19 TRANSFORMATION OF NEWTON'S EQUATIONS

Suppose we have two charges, e, one fixed at the origin, and the other at the point $(0, y, 0)$ of a coordinate system at rest with respect to us. The equation of motion of the second charge is then

$$m\ddot{y} = \frac{e^2}{y^2} \tag{19.1}$$

Next consider the case when the charges are moving with velocity v, (uniform) with respect to us. In section 11 we saw that the field in the equatorial plane of a uniformly moving point charge is

$$E_\perp = \frac{e}{\sqrt{1 - \dfrac{v^2}{c^2}}(y^2 + z^2)}$$

Now the force on one due to the magnetic field of the other is

$$e\left[\frac{\mathbf{v}}{c} \times \mathbf{H}\right] = e[\boldsymbol{\beta} \times (\boldsymbol{\beta} \times \mathbf{E})]$$

by (11.7) and hence

$$= e\{(\boldsymbol{\beta} \cdot \mathbf{E})\,\boldsymbol{\beta} - \beta^2\mathbf{E}\}$$

Thus for two particles moving together in a direction perpendicular to their line of centers, the magnetic force is

$$-e\beta^2 E_\perp$$

Hence the total force acting on one particle is

$$\begin{aligned} F &= e(1 - \beta^2)\, E_\perp \\ &= \frac{e^2}{y^2}\sqrt{1 - \beta^2} \end{aligned}$$

and the equation of motion is

$$m\ddot{y} = \frac{e^2}{y^2}\sqrt{1 - \beta^2} \tag{19.2}$$

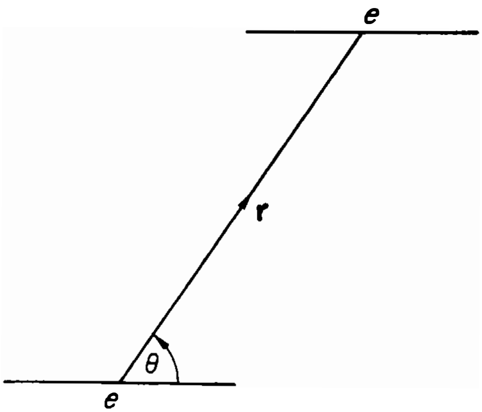
Let us now follow the charges so that they seem to be at rest with respect to us. We would expect to get the equations of motion in this new situation by making the usual transformation

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z &= z \\ t' &= t \end{aligned} \tag{19.3}$$

on (19.2). The result is

$$m\ddot{y}' = \frac{e^2}{y'^2}\sqrt{1 - \beta^2} \tag{19.4}$$

This is different from (19.1) and it means that the phenomenon depends on the state of motion of the system with respect to a certain fixed system. It is important to note that the Eqs. (19.1) and (19.4) differ by terms in the second order of $\beta = v/c$. Since in most natural phenomena $v/c \ll 1$, it is useful to classify terms in the powers of v/c.



Next suppose that the charges are staggered so that the line joining them is not perpendicular to the line of motion. Then

$$\mathbf{E} = \frac{e(1 - \beta^2)\,\mathbf{r}}{r^3(1 - \beta^2 \sin^2\theta)^{3/2}}$$

$$\mathbf{H} = \boldsymbol{\beta} \times \mathbf{E}$$

$$\mathbf{F} = e\{\mathbf{E} + \boldsymbol{\beta} \times \mathbf{H}\}$$
$$= \frac{e^2(1 - \beta^2)}{r^3(1 - \beta^2 \sin^2 \theta)^{3/2}} \{\mathbf{r}(1 - \beta^2) + (\boldsymbol{\beta} \cdot \mathbf{r}) \boldsymbol{\beta}\} \tag{19.5}$$

which shows that the direction of the force is not along the radius vector, and for $v \to c$, the direction approaches that of the velocity. Thus, if the simple transformation laws are correct, we could find, by observing the behavior of charges, how fast and in what direction we were moving with respect to the preferred reference frame in which the phenomenon takes a simple form.

Thus we are led to one of three possible conclusions:

1) The above prediction is correct, and we can find a system in which the equation of motion takes a simple form. According to this theory the earth cannot be moving with a very high velocity with respect to the preferred system, since we do not observe the complications predicted above.

2) There is something wrong in the calculation of the field made above. This means that Maxwell's equations hold only in a certain coordinate system.

3) The transformation equations used in going from one coordinate system to another moving with respect to it are not correct.

In considering the validity of Maxwell's equations, we need not consider the whole set but only the wave equation which $\mathbf{E}$ and $\mathbf{H}$ satisfy as a consequence of Maxwell's equations:

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \Delta \mathbf{E} = 0$$

This equation does not transform to a similar equation in the primed variables under the transformation (19.3), which is called the Galilean transformation. It is due to the fact that the velocity of light, c, is not constant for all systems under such a transformation. The reason for choosing the Galilean transformation in the first place is that under such a transformation Newton's equation

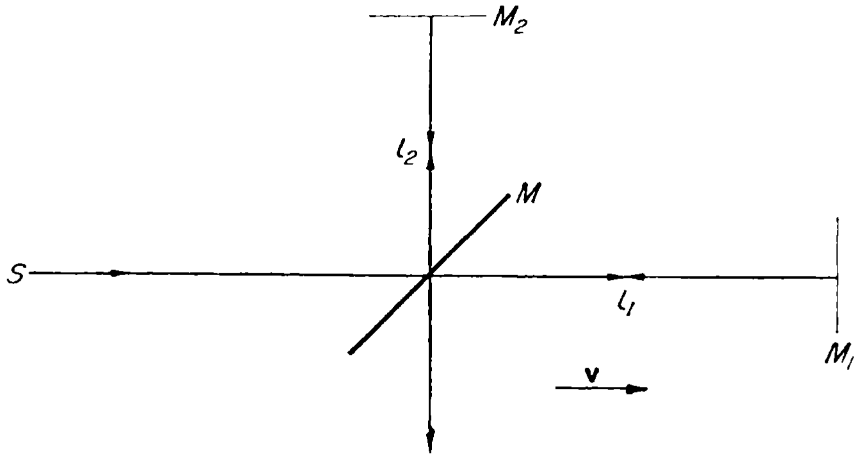
$$m\ddot{\mathbf{x}} = \mathbf{F}$$

retains its form in all systems. Thus, if we rule out the first possibility, we have to make a choice whether to keep Maxwell's or Newton's equations as holding universally. We shall find that Newton's equation has to be modified.

20 MICHELSON–MORLEY AND KENNEDY–THORNDYKE EXPERIMENTS

The possibility of the earth having an absolute motion with respect to the rest system in which the equations describing natural phenomena take a simple form was first tested experimentally by Michelson and Morley. They argued that if the earth has an absolute motion, the velocity of light would be different in different directions, and hence there should be a shift in the fringes of an interferometer when it is rotated through 90°.

The figure shows the schematic layout of an interferometer. M_1 and M_2 are mirrors at the ends of two arms, and M is a half-silvered mirror which



splits the light coming from S. Suppose the arm (1) is parallel to the direction of the absolute velocity v; then the time taken for the light to go from M to M_1 and back to M is

$$t_{\parallel}^{(1)} = \frac{l_1}{c-v} + \frac{l_1}{c+v}$$

$$= \frac{2l_1}{c}\,\frac{1}{1-\beta^2} \tag{20.1}$$

The corresponding time for the arm (2) is

$$t_{\perp}^{(2)} = \frac{2l_2}{c\sqrt{1-\beta^2}} \tag{20.2}$$

Now if the system is rotated through 90⁰ so that arm (2) is parallel and arm (1) is perpendicular to the absolute velocity, the corresponding times are

$$t_{\parallel}^{(2)} = \frac{2l_2}{c}\frac{1}{1-\beta^2} \tag{20.3}$$

$$t_{\perp}^{(1)} = \frac{2l_1}{c}\frac{1}{\sqrt{1-\beta^2}} \tag{20.4}$$

Hence

$$\begin{aligned}\Delta T &\equiv (t_{\parallel}^{(1)} - t_{\perp}^{(2)}) - (t_{\perp}^{(1)} - t_{\parallel}^{(2)})\\ &= \frac{2l_1}{c}\left(\frac{1}{1-\beta^2} - \frac{1}{\sqrt{1-\beta^2}}\right) + \frac{2l_2}{c}\left(\frac{1}{1-\beta^2} - \frac{1}{\sqrt{1-\beta^2}}\right)\\ &= \frac{2(l_1+l_2)}{c}\left(\frac{1}{1-\beta^2} - \frac{1}{\sqrt{1-\beta^2}}\right)\end{aligned} \tag{20.5}$$

For $l_1 = l_2 = l$ and $\beta \ll 1$, we have

$$\Delta T \approx \frac{2l}{c}\beta^2 \tag{20.6}$$

The experiment performed by Michelson and Morley was accurate enough to detect a few tenths of a percent of the expected fringe shift if for v the velocity of the earth in its orbit was taken. However no shift was observed.

To explain the null result, Fitzgerald and Lorentz advanced the contraction hypothesis which states that all lengths parallel to the direction of the absolute velocity contract in the ratio

$$l_{\parallel} = \sqrt{1-\beta^2}\,l_{\perp}$$

which is the same as a contraction in the ratio of $1 : \sqrt{1-\beta^2}$ between the length at rest and in motion.

If we put $l_1 = l$, $l_2 = l + a$, then including the contraction, we have

$$t_{\parallel}^{(1)} = \frac{2l}{c\sqrt{1-\beta^2}}$$

$$t_{\perp}^{(2)} = \frac{2(l+a)}{c\sqrt{1-\beta^2}}$$

$$t_{\parallel}^{(1)} - t_{\perp}^{(2)} = \frac{2a}{c\sqrt{1-\beta^2}} \tag{20.7}$$

Thus since the velocity of the earth in its orbit should make $\mathbf{v}$ vary with time, we would expect a gradual shift during a year in the fringes of a stationary interferometer. This experiment was done by Kennedy and Thorndyke, and it also led to a null result. This observation can be explained, if, in addition to the contraction in length, we have a dilation in time:

$$\Delta t = \frac{\Delta \tau}{\sqrt{1-\beta^2}}$$

where $\Delta \tau$ is a time interval recorded by a clock which moves along with the interferometer, and Δt is the corresponding interval recorded by a clock which is at rest with respect to the absolute frame. Since the wavelength of the light used in the interferometer is determined by atomic clocks (i.e., excited atoms) which move with the interferometer, Eq. (20.7) should under the time dilation hypothesis, be replaced by

$$\tau_{\parallel}^{(1)} - \tau_{\perp}^{(2)} = \frac{2a}{c} = \text{independent of } \beta$$

[The length contraction hypothesis may be made plausible by refering to Eq. (19.5) which gives the force between two moving charges. In this equation $\mathbf{r}$ is the separation vector between the two charges as measured in the absolute frame. If the length contraction hypothesis is correct then the separation vector as measured in a frame (of actual physical meter sticks) moving with the charges is given by

$$\mathbf{r}' = \mathbf{T} \cdot \mathbf{r}$$

where

$$\mathbf{T} = \mathbf{1} + \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)\frac{\boldsymbol{\beta}\boldsymbol{\beta}}{\beta^2}$$

$$\mathbf{T}^{-1} = \mathbf{1} - (1 - \sqrt{1-\beta^2})\ \frac{\boldsymbol{\beta}\boldsymbol{\beta}}{\beta^2}$$

$$\mathbf{T}^2 = \mathbf{1} + \frac{1}{1-\beta^2}\boldsymbol{\beta}\boldsymbol{\beta}$$

The force measured in the moving frame may be determined by appeal to the energy principle, which should remain valid. We must have

$$\mathbf{F}' \cdot d\mathbf{r}' = \mathbf{F} \cdot d\mathbf{r}$$

and hence

$$\mathbf{F}' = \mathbf{T}^{-1} \cdot \mathbf{F}$$

By direct substitution of expression (19.5) one finds

$$\mathbf{F}' = \sqrt{1 - \beta^2}\,\frac{e^2\mathbf{r}'}{r'^3}$$

which, except for the factor $\sqrt{1 - \beta^2}$, has the same form as the electrostatic force between two charges at rest. This means that for every dynamical configuration of a collection of slow-moving charges, the center of gravity of which is at rest, there exists another physically realizable configuration in which the center of gravity moves with (arbitrary) velocity $\mathbf{v}$ and the configuration as a whole is contracted, in the direction of $\mathbf{v}$, by the factor $\sqrt{1 - \beta^2}$. Lorentz reasoned that since matter is made up of electric charges, all bodies must show this contraction.

It will be noted that although the configuration suffers the Lorentz contraction, the orbital motion is slowed down because of the factor $\sqrt{1 - \beta^2}$ in the expression for $\mathbf{F}'$. This is in the right direction to produce the time dilation effect, but does not account for all of it. The full effect is obtained by postulating, in addition, an increase in all masses by the factor $1/\sqrt{1 - \beta^2}$. Ed.]

21 LORENTZ TRANSFORMATION

The time dilation can be shown to follow from the Lorentz contraction and the principle of relativity. We shall assume not only that the length of a rod in a moving system is contracted, but also that the length of a rod in a rest system appears contracted to an observer in a moving system. This assumption of the relativity of the Lorentz contraction has far reaching consequences on our ideas of simultaneity.

The length of a rod is obtained by finding the coordinates of its end points at a certain instant t. For a coordinate system in which the rod is at rest we have

$$l = x_1(t) - x_2(t)$$

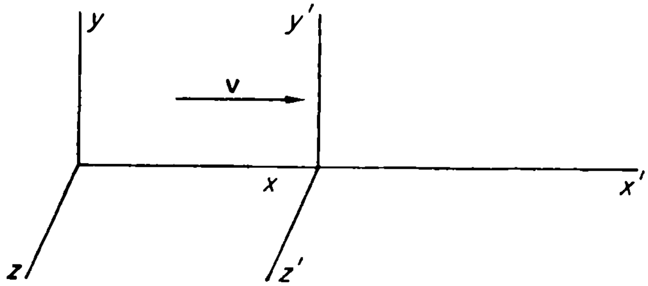
For a system in which it is moving,

$$l' = x_1'(t') - x_2'(t')$$

and

$$l' \neq l$$

Suppose there is another parallel rod, of equal proper length, which is at rest in the primed system. Consider the instant when the left hand ends of the rods are in coincidence (see figure). Then the right hand ends will not be in coincidence in either frame. An observer in the unprimed system sees that the right end of the rod moving with respect to him gives a reading of $l\sqrt{1 - \beta^2}$, as measured by his own rod, at the same instant that he observes the left ends to be in coincidence. For an observer in the primed system, however, these observations are not simultaneous since when he left ends of the rods are in coincidence for him, the right end of his own rod extends beyond the right end of the rod moving with respect to him.



Consider two coordinate systems, one moving with respect to the other with velocity v along the x axis. From the Lorentz contraction we have

$$\sqrt{1 - \beta^2}x' = x - vt \tag{21.1}$$

and from the condition of relativity

$$\sqrt{1 - \beta^2}x = x' + vt'$$

Therefore

$$\sqrt{1 - \beta^2}x = \frac{x - vt}{\sqrt{1 - \beta^2}} + vt'$$

and

$$t' = \frac{t - \dfrac{v}{c^2}x}{\sqrt{1 - \beta^2}} \tag{21.2}$$

The inverse transformation can be obtained simply by changing the sign of v, as this is just the condition of relativity. Thus

$$t = \frac{t' + \dfrac{v}{c^2}x'}{\sqrt{1 - \beta^2}}$$

We can easily verify this by eliminating x in the first transformation equation. Writing

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

we have

$$t' = \gamma\left[t - \frac{v}{c^2}\gamma(x' + vt')\right]$$

$$\left(1 - \frac{\beta^2}{1-\beta^2}\right)t' = \gamma t - \frac{v}{c^2}\gamma^2 x'$$

$$t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

This explains the null result in the Kennedy-Thorndyke experiment, since for $x' = 0$

$$t = \frac{t'}{\sqrt{1-\beta^2}}$$

We have so far obtained the transformation equations for x and t. The Michelson–Morley experiment also tells us that the length perpendicular to the direction of motion must be unaltered. Hence

$$y' = y \tag{21.3}$$

$$z' = z \tag{21.4}$$

The transformations (21.1) to (21.4) constitute the Lorentz transformation. This transformation can be obtained from two general postulates:

1) complete relativity
2) constancy of the velocity of light

This method of derivation is first due to Poincaré. The second postulate states that for two coordinate systems, one moving relative to the other with uniform velocity $\mathbf{v}$, if

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = c^2t^2$$

then

$$(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2 = c^2t'^2$$

and vice versa. By a proper choice of the origins of the coordinate system, we can write

$$x_2 - x_1 = x$$

$$x'_2 - x'_1 = x'$$

Thus we can write

$$x^2 + y^2 + z^2 - c^2t^2 = K^2(x'^2 + y'^2 + z'^2 - c^2t'^2) \tag{21.5}$$

etc. We want to find the most general transformation satisfying (21.5) and compatible with the first postulate. The transformation must be linear, since if it were not, there would arise singularities in space and this is not permissible; and also a uniform motion in one system would not correspond to a uniform motion in another system.

EXERCISE 25 Prove that in order that a uniform motion in one system correspond to a uniform motion in another, the transformation must be linear.

Further, for a linear function of x, y, z, t to correspond to a linear function of x', y', z', t', K cannot be a function of x, y, z or t. If K is a function of $\mathbf{v}$ then due to the principle of relativity, it can only depend on the absolute magnitude of $\mathbf{v}$. Moreover, each transformation must have an inverse. That is, the product of a transformation to a system with velocity $\mathbf{v}$ and a transformation to a system with velocity $-\mathbf{v}$ must reduce to the identity transformation. This implies

$$K^2(|\mathbf{v}|)\,K^2(|-\mathbf{v}|) = 1$$

whence

$$K^2 = 1$$

For a relative velocity along the x axis, the most general transformation possible under the above conditions is

$$\begin{aligned} x' &= Ax + Bt \\ t' &= Cx + Dt \\ y' &= Ey \\ z' &= Fz \end{aligned} \tag{21.6}$$

Now if we take $x = t = z = 0$ in (21.5) then

$$y^2 = y'^2$$

There is no reason why the direction perpendicular to the velocity should change. If it did, the first postulate would be violated. Hence

$$E = \pm 1, \quad \text{and similarly} \quad F = \pm 1$$

We shall take the positive signs, since the negative signs correspond to reflection of the coordinate axes. Substituting in (21.5) we have

$$A^2x^2 + 2ABxt + B^2t^2 - c^2(C^2x^2 + 2CDxt + D^2t^2) = x^2c^2t^2$$

$$A^2 - c^2C^2 = 1$$

$$AB - c^2CD = 0$$

$$B^2 - c^2D^2 = -c^2$$

and in addition we have

$$\frac{B}{A} = -v$$

if the primed coordinate is moving with velocity v along the positive x direction of the unprimed coordinate. Thus we have four equations for four unknowns and the solution of these equations is

$$A = \gamma$$

$$B = -v\gamma$$

$$C = -\frac{v}{c^2}\gamma$$

$$D = \gamma$$

where

$$\gamma = 1/\sqrt{1 - \beta^2}$$

With these values, (21.6) are identical with the Eqs. (21.1) to (21.4). They can be written in vector form thus:

$$\mathbf{r}' = \gamma\{(\mathbf{n} \cdot \mathbf{r})\,\mathbf{n} - \mathbf{v}t\} + \{\mathbf{r} - (\mathbf{n} \cdot \mathbf{r})\,\mathbf{n}\}$$

$$t' = \gamma\left(t - \frac{\mathbf{v} \cdot \mathbf{r}}{c^2}\right)$$

where $\mathbf{n}$ is a unit vector in the direction of $\mathbf{v}$.

Einstein gave a physical interpretation of the Lorentz contraction by studying the sychronization of clocks. Two clocks, A and B, at rest with respect to each other and near together can be said to be synchronized if they read the same. However, if they are not close together, the two clocks cannot be observed at once. We must use some kind of a signal. Since the velocity of light is constant in all frames, and hence reliable, we use light. A signal sent from clock A, when A reads t_1, reaches clock B when B reads t_2, whereupon it is partially reflected and returns to A when A reads t_3. The clocks are synchronized if

$$t_3 - t_2 = t_2 - t_1$$

or

$$t_2 = \frac{1}{2}(t_1 + t_3)$$

It is clear however, that due to the constancy of the velocity of light, two clocks synchronized for an observer at rest with respect to them, will not be synchronized for a moving observer.

The time dilation gives rise to a "paradox". Suppose we have a set of identical twins A and B. B is taken on a long journey in a straight line with uniform velocity (except for short intervals of time when he is accelerated to attain this motion from rest, to reverse his velocity, and to return to rest beside A). On return, B will be younger than A, since B's clock has been ticking more slowly than A's. By relativity, B may be tempted to say that it was A who has been in motion and not B so that A should be the younger. However, relativity does not apply since there is an asymmetry in the problem. B has undergone acceleration, whereas A has not. This suffices to resolve the "paradox," although one should be cautioned *not* to assume that the difference in their ages is a function solely of the duration and magnitude of B's acceleration. It depends also on the duration of the trip.

Next let us consider the transformation of velocities. As before, let the primed system move with velocity $\mathbf{v}$ along the x axis of the unprimed system. We know that, if terms of the order v^2/c^2 and higher are neglected, we get

$$V_x = V'_x + v$$

where

$$V_x = \frac{dx}{dt}, \qquad V'_x = \frac{dx'}{dt'}$$

Now from (21.1) we have

$$\begin{aligned} dx' &= \gamma(dx - v\,dt) \\ &= \gamma(V_x - v)\,dt \end{aligned}$$

and from (21.2)

$$dt' = \gamma\left(dt - \frac{v}{c^2}dx\right)$$

$$= \gamma\left(1 - \frac{vV_x}{c^2}\right)dt$$

Therefore

$$V'_x = \frac{V_x - v}{1 - \dfrac{vV_x}{c^2}} \tag{21.7}$$

and the equation for V_x is

$$V_x = \frac{V'_x + v}{1 + \dfrac{vV'_x}{c^2}} \tag{21.8}$$

This shows that the sum of any two velocities is less than c unless either one of them equals c. This is consistent with our postulate that the velocity of light is c in all systems.

From (21.2) and (21.3) we also have

$$dt' = \gamma\left(dt - \frac{v}{c^2}dx\right)$$

$$dy' = dy$$

$$V'_y = \frac{V_y}{\gamma\left(1 - \dfrac{vV_x}{c^2}\right)} \tag{21.9}$$

and the inverse transformation is

$$V_y = \frac{V'_y}{\gamma\left(1 + \dfrac{vV'_x}{c^2}\right)} \tag{21.10}$$

Similarly from (21.2) and (21.4) we get

$$V'_z = \frac{V_z}{\gamma\left(1 - \dfrac{vV_x}{c^2}\right)} \tag{21.11}$$

$$V_z = \frac{V'_z}{\gamma\left(1 + \dfrac{vV'_x}{c^2}\right)} \tag{21.12}$$

EXERCISE 26 Suppose a meson with mass 200 m moving with velocity $v = 0.95c$ breaks up into two particles, each of mass m. By using the conservation laws of energy and momentum, calculate the velocities of the two resultant particles and show that the same values can be obtained by using a reference system in which the meson is originally at rest and then using the transformation equations for the velocities. (Use the following expressions for energy and momentum:

$$E = \frac{mc^2}{\sqrt{1 - \beta^2}}$$

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \beta^2}}$$

See section 28.)

22 MINKOWSKI DIAGRAM

It is possible to give a graphical representation of the length contraction and the time dilation. Let $l = ct$; then the Lorentz transformation (21.1) to (21.4) can be written

$$x' = \gamma x - \beta\gamma l$$

$$l' = \gamma l - \beta\gamma x$$

where

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

so that

$$\gamma^2 = \gamma^2\beta^2 + 1$$

Hence if we let

$$\gamma = \cosh q$$

then

$$\beta\gamma = \sinh q$$

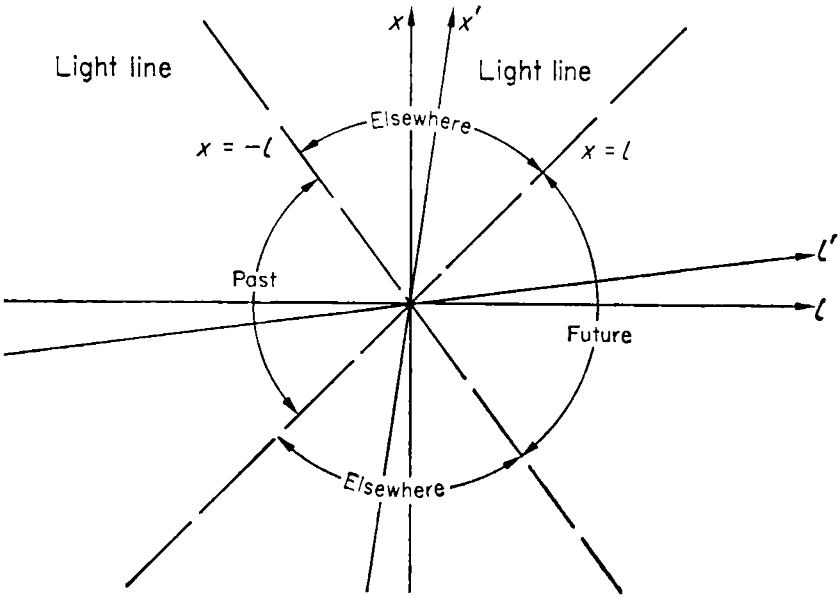
$$\beta = \tanh q$$

and

$$x' = x \cosh q - l \sinh q$$

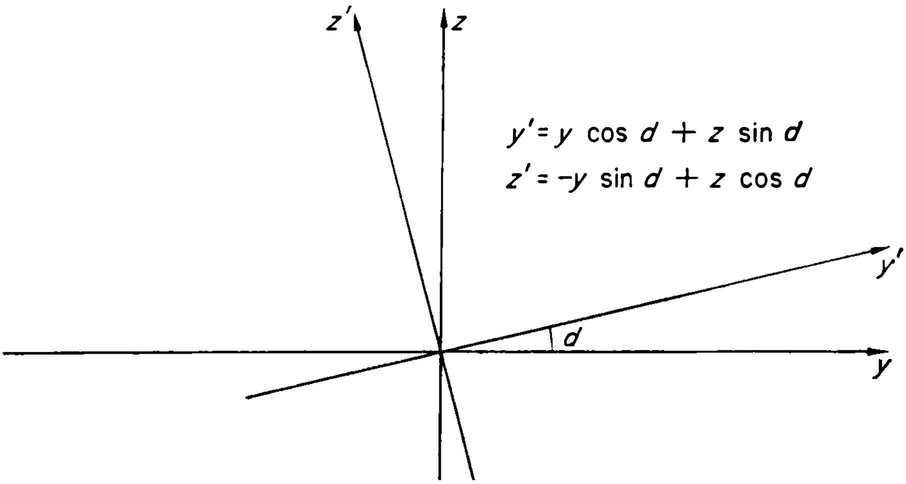
$$l' = -x \sinh q + l \cosh q$$

Thus the axes x' and l' represented in the x, l-plane, have the following shape:

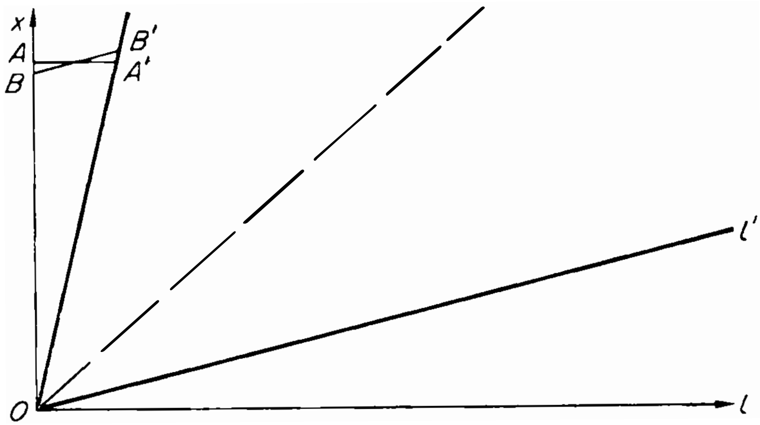


The paths of light rays passing through the origin are the lines $x = \pm l$ and these lines divide the plane into four regions which for an observer at rest at the origin of the unprimed system represent future, past, and elsewhere. The regions called elsewhere are so named since any disturbance originating in these regions cannot affect the observer due to the fact that no signal can travel faster than light. This representation is called the Minkowski diagram.

It is interesting to compare the above transformation with that of the rotation of space axes in a plane



The Lorentz contraction can be represented in the Minkowski diagram. As usual, let the primed system, C', move with velocity v along the x axis of the unprimed system, C. Suppose we have a measuring rod of unit length at rest in C, with one end at the origin, and the other at A. The



world line of the first end is the l axis, and that of the other and is a line parallel to the l axis and cuts the x' axis at A'. We want to show that $0A'$, which is the length of the rod as seen by an observer in C', is of length $\sqrt{1-\beta^2}$. We cannot compare the lengths of the lines $0A$ and $0A'$ in the diagram, since the invariant in this case is

$$x^2 - l^2$$

and not the sum of the squares of the coordinates, $y^2 + z^2$, say, as in the case of space rotation. We have to find the coordinate x' of A' in terms of the coordinates x and l of C.

Now the equation of the x' axis in C is

$$l = \beta x$$

and hence for A' we have $l' = 0$, $l = \beta x$. The invariant relation

$$x'^2 - l'^2 = x^2 - l^2$$

thus reduces to

$$x'^2 = x^2(1 - \beta^2)$$

and since we also have $x = 1$ for A and A', we obtain

$$x' = \sqrt{1 - \beta^2}$$

which is just the Lorentz contraction.

It can be shown similarly that a rod at rest in C' appears contracted in C. Let $0B'$ be the length of a unit rod at rest in C. Then the world line of B' will intersect the x axis at B, and we have for it $l = 0$, $l' = -\beta x'$, $x' = 1$, so that the invariant gives

$$x^2 = 1 - \beta^2$$

and hence the Lorentz contraction.

EXERCISE 27 Show how the time dilation can be illustrated in the Minkowski diagram.

23 DERIVATION OF THE FRESNEL COEFFICIENT AND THE ABERRATION FORMULA

We shall give here two applications of the velocity addition formula to the explanation of two experiments which were important in the development of the theory of relativity. First let us consider Fizeau's experiment. Here the times required for light to travel through water with and against its direction of motion are compared. If the water is at rest, the velocity in it is simply c/n where n is the index of refraction of water. If the water has velocity v, then by (21.8) the light travelling with it has the velocity

$$\frac{v + \dfrac{c}{n}}{1 + \dfrac{v}{cn}} = \frac{c}{n}\left(\frac{1 + \dfrac{nv}{c}}{1 + \dfrac{v}{cn}}\right)$$

$$\approx \frac{c}{n}\left[1 + \frac{v}{c}\left(n - \frac{1}{n}\right)\right] \tag{23.1}$$

for $v/c \ll 1$. Thus relativity gives a simple explanation of the Fresnel dragging coefficient.

Next let us consider aberration. Let the reference system C' move with velocity v along the x axis of system C. Suppose a particle has velocity V' making an angle with the x' axis in C'. We want to calculate these quantities in the unprimed system. From (21.8) we have

$$V_x = \frac{V'_x + v}{1 + \dfrac{vV'_x}{c^2}}$$

$$= \frac{V' \cos\alpha' + v}{1 + \dfrac{vV' \cos\alpha'}{c^2}}$$

and from (21.10)

$$V_y = \frac{V'_y}{\gamma\left(1 + \dfrac{vV'_x}{c^2}\right)}$$

$$= \frac{V' \sin \alpha'}{\gamma\left(1 + \dfrac{vV' \cos \alpha'}{c^2}\right)}$$

Hence

$$\tan \alpha = \frac{V_y}{V_x} = \frac{V' \sin \alpha'}{\gamma(V' \cos \alpha' + v)}$$

$$= \frac{\sqrt{1 - \beta^2} \tan \alpha'}{1 + \dfrac{v}{V'} \sec \alpha'} \tag{23.2}$$

For the case of light we have $V' = c$ and

$$\tan \alpha = \frac{\sqrt{1 - \beta^2} \tan \alpha'}{1 + \dfrac{v}{c} \sec \alpha'} \tag{23.3}$$

This is the correct aberration formula which checks with all the experiments.

It seems a bit remarkable that we can obtain correct formulae by simple Lorentz transformation when we consider the complexity of the problem, as for instance if we try to obtain the dispersion formula when all the electrons are moving uniformly in addition to their harmonic motion. The reason we can derive a formula in a coordinate system which is most convenient for the calculation and then make a Lorentz transformation to get the result in the required system is due to the fact that true physical equations hold in all systems. That is, the equations derived that describe natural phenomena are invariant under Lorentz transformation.

24 COVARIANCE

Let q_i stand for any one of the observables **r**, t, **E**, **H**, m, **j**, etc. in the coordinate system C with which we are concerned. There is some relationship of the form

$$A_j(q) = 0$$

which expresses a physical law. Let the corresponding value of q in another coordinate system C' be q'. C and C' are connected by a Lorentz transformation. The same law will be expressed in C' by the equation

$$A_k'(q') = 0$$

In general

$$q' \neq q$$

and A' is a different function of q' from A of q. The idea of covariance is expressed in the relation

$$A_j(q) = 0 \rightleftharpoons A_k'(q') = 0$$

This condition is equivalent to the relation

$$A_k' = \sum_j Q_{kj} A_j$$

where $|Q_{kj}| \neq 0$, since the vanishing of one set implies the vanishing of the other. If the above relation holds for a Lorentz transformation then the equation

$$A_i(q) = 0$$

is said to be Lorentz covariant.

As an illustration consider one of Maxwell's equations

$$\nabla \cdot \mathbf{E} - 4\pi\varrho = 0$$

If we make a Lorentz transformation, we will not get

$$\nabla \cdot \mathbf{E}' - 4\pi\varrho' = 0$$

but some more complicated expression. This is because the equation is only part of a general law. The other part is

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - 4\pi\mathbf{j} = 0$$

and the correct relation is

$$\nabla \cdot \mathbf{E}' - 4\pi\varrho' = q_1(\nabla \cdot \mathbf{E} - 4\pi\varrho) + p_2 \cdot \left(\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - 4\pi\mathbf{j}\right)$$

where q_1 and p_2 are certain scalar and vector constants respectively.

We have seen that the Lorentz transformation mixes up the time and the space coordinates. Thus

$$(x', y', z', t') = Q(x, y, z, t)$$

and we found that Q, the matrix of the transformation can be written in the form

$$Q = \begin{array}{c|cccc} & x & y & z & ct \\ \hline x' & \gamma & 0 & 0 & -\dfrac{v}{c}\gamma \\ y' & 0 & 1 & 0 & 0 \\ z' & 0 & 0 & 1 & 0 \\ ct' & -\dfrac{v}{c}\gamma & 0 & 0 & \gamma \end{array} \tag{24.1}$$

If we let x^1, x^2, x^3, x^4 stand for x, y, z, ct, then

$$(x^\mu)' = Q^\mu_\nu x^\nu$$

where μ stands for any one of 1, 2, 3, 4, and we are using the summation convention that two repeated indices, one upper and one lower, on the same side of the equation are to be summed from one to four. Q^μ_ν is the element in the μth row and νth column of the matrix Q given above.

Any set of four quantities A^μ which transform according to the law

$$(A^\mu)' = Q^\mu_\nu A^\nu \tag{24.2}$$

is called a contravariant 4-vector. If x_μ stands for $x^1, x^2, x^3, -x^4$, for $\mu = 1, 2, 3, 4$, respectively, then x_μ is said to form a covariant 4-vector. Now

$$\begin{aligned} x^\mu x_\mu &= x^2 + y^2 + z^2 - c^2t^2 \\ &= \text{invariant} \end{aligned}$$

An invariant is a scalar quantity which remains unaltered during any transformation of the coordinate system. A set of 4 quantities B_μ which transform like x_μ is called a covariant 4-vector. It is clear that

$$A^\mu B_\mu = \text{invariant}$$

The set of 16 quantities $A^\mu B^\nu$ transform according to the law

$$(A^\mu B^\nu)' = Q^\mu_\lambda Q^\nu_\gamma A^\lambda B^\gamma \tag{24.3}$$

and is said to form a tensor of second rank. In the same way, a tensor of arbitrarily high rank can be formed by taking the product of vectors thus

$$A^\mu B^\nu C_\lambda D^\eta$$

and they transform according to the law

$$(A^{\mu}B^{\nu}C_{\lambda}D^{\eta}\ldots)' = Q^{\mu}_{\alpha}Q^{\nu}_{\beta}Q^{\gamma}_{\lambda}Q^{\eta}_{\delta}\ldots A^{\alpha}B^{\beta}C_{\gamma}D^{\delta}\ldots \tag{24.4}$$

Such a complicated set of quantities does not usually occur in nature, but symmetric and antisymmetric tensors of second rank are frequent. The former is such that each element is unaltered by the interchange of the indices μ, ν, and so has 10 independent components; the latter is such that each element merely changes sign by the interchange of μ and ν and has 6 independent components. In classical mechanics, the Q^{μ}_{ν} are uniquely determined since they connect observable quantities; in quantum mechanics, however, since the wave function ψ_1 is not an observable, but only the square of its absolute value is, the sign of its transformation coefficients is undetermined.

The values of Q^{μ}_{ν} are greatly restricted since the Lorentz transformations form a group. That is, the sucessive application of two transformations L and L' must be equivalent to another Lorentz transformation L''. Thus

$$\begin{aligned}(x^{\mu})'' &= \overset{L''}{Q^{\mu}_{\nu}}x^{\nu}\\ &= \overset{L'}{Q^{\mu}_{\alpha}}(x^{\alpha})'\\ &= \overset{L'}{Q^{\mu}_{\alpha}}\overset{L}{Q^{\alpha}_{\nu}}x^{\nu}\end{aligned}$$

and therefore

$$\overset{L''}{Q^{\mu}_{\nu}} = \overset{L'}{Q^{\mu}_{\alpha}}\overset{L}{Q^{\alpha}_{\nu}} \tag{24.5}$$

It turns out that all quantities arising in electromagnetic theory transform according to laws involving the matrix Q^{μ}_{ν} or product of Q's. A physical law may be written in covariant form, but its validity must be tested by experiment.

Consider a general tensor of second rank

$$T^{\mu\nu} = A^{\mu}B^{\nu}$$

We can write it as a sum of 3 parts which are themselves tensors. First we can separate it into the symmetric and antisymmetric components

$$\begin{aligned}T^{\mu\nu} &= \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu}) + \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu})\\ &= S^{\mu\nu} + A^{\mu\nu}\end{aligned}$$

$S^{\mu\nu}$ has 10 and $A^{\mu\nu}$ has 6 independent components. This separation is preserved under a Lorentz transformation since the transformation matrix

$$Q^{\mu}_{\alpha} Q^{\nu}_{\beta}$$

is invariant under the simultaneous interchanges $\mu \rightleftarrows \nu$ and $\alpha \rightleftarrows \beta$. We can further write

$$S^{\mu\nu} = \frac{1}{4} \delta^{\mu\nu} S^{\alpha}_{\alpha} + \tilde{S}^{\mu\nu}$$

where

$$(\delta^{\mu\nu}) = (\delta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$(\delta_{\mu\sigma} \delta^{\sigma\nu}) = (\delta^{\nu}_{\mu}) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$S^{\beta}_{\alpha} = \delta_{\alpha\nu} S^{\nu\beta}$$

This separation is also invariant under Lorentz transformations. Thus

$$T^{\mu\nu} = \tilde{S}^{\mu\nu} + A^{\mu\nu} + \frac{1}{4} \delta^{\mu\nu} S^{\alpha}_{\alpha}$$

where $\tilde{S}^{\mu\nu}$ has 9 independent components, $A^{\mu\nu}$ has 6, and $\delta^{\mu\nu} S^{\alpha}_{\alpha}$ has one. The antisymmetric part $A^{\mu\nu}$ can be thought of as a 4-dimensional generalization of the vector product since 4-vectors B^{μ} and C^{μ} may always be found such that

$$A^{\mu\nu} = B^{\mu} C^{\nu} - C^{\mu} B^{\nu}$$

It may also be regarded as a surface vector, since the six independent components are the projections on the six coordinate planes of the area determined by B^{μ} and C^{μ}.

Another type of tensor which occasionally occurs is a completely antisymmetrical tensor of the third rank, $T^{\mu\nu\lambda}$. The components of such a tensor vanish unless all three indices are different. Therefore it has only 4 independent components, and they can be characterized by the missing index.

We shall later make use of the operator ∂_μ defined by

$$(\partial_\mu) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{1}{c}\frac{\partial}{\partial t}\right)$$

EXERCISE 28 Prove that ∂_μ is a covariant 4-vector. The contravariant form of ∂_μ is defined by

$$\partial^\mu = \delta^{\mu\nu}\,\delta_\nu, \qquad (\partial^\mu) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, -\frac{1}{c}\frac{\partial}{\partial t}\right)$$

From the two together we can construct the following scalar operator:

$$\square \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

25 TRANSFORMATION LAWS OF ELECTROMAGNETIC QUANTITIES

Let us now apply the ideas developed in the last section to write Maxwell's equations in covariant form and to see how the quantities which occur in it transform under a Lorentz transformation. We have

$$\nabla \cdot \mathbf{H} = 0 \tag{25.1}$$

$$\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{25.2}$$

$$\nabla \cdot \mathbf{E} = 4\pi\varrho \tag{25.3}$$

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 4\pi\mathbf{j} \tag{25.4}$$

Let us first consider (25.3) and (25.4). We know that (ϱ, j) must transform to (ϱ', j'), and that they must do so in such a way that, if the charge is at rest in the unprimed system (that is $\mathbf{j} = 0$) then

$$\varrho' \approx \varrho_0$$

$$\mathbf{j}' \approx \frac{\varrho_0 \mathbf{v}}{c}$$

to the first order in v/c. We are using the subscript 0 to indicate quantities measured in the rest system. Now the total charge $\int \varrho\, dV$ must also be an invariant. We would meet with great difficulties if this were not so. Since the volume contracts by the factor $\sqrt{1 - v^2/c^2}$, we must have

$$\varrho' = \frac{\varrho_0}{\sqrt{1 - v^2/c^2}}$$

Thus ϱ transforms like t, and so we are led to make the guess that $(\mathbf{j}, \varrho)$ form a contravariant 4-vector j^μ. Then

$$(j^1)' = (j_x)' = \gamma j_x + \beta\gamma\varrho$$

$$(j^2)' = (j_y)' = j_y$$

$$(j^3)' = (j_z)' = j_z$$

$$(j^4)' = \varrho' = \gamma\varrho + \beta\gamma j_x$$

This transformation law guarantees both the invariance of the total charge and the reduction to correct values for low velocities.

The same argument may be given in another way. $(\mathbf{j}, \varrho)$ satisfy a conservation law

$$\frac{1}{c}\frac{\partial\varrho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{25.5}$$

which must be covariant under Lorentz transformations, and this is the case if $(\mathbf{j}, \varrho)$ form a contravariant vector since then (25.5) can be written as

$$\partial_\mu j^\mu = 0 \tag{25.6}$$

Thus the right sides of (25.3) and (25.4) form a contravariant 4-vector. The left sides have differential operators ∂_μ, so that the simplest way of writing the equations in covariant form is

$$\partial_\alpha F^{\mu\alpha} = 4\pi j^\mu \tag{25.7}$$

Since only 6 quantities $\mathbf{E}$, $\mathbf{H}$ occur on the left, we suspect $F_{\alpha\mu}$ to be an antisymmetric tensor. To find $F_{\alpha\mu}$ in terms of $\mathbf{E}$, $\mathbf{H}$ let us introduce the potentials $\mathbf{A}$ and φ such that

$$\mathbf{H} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$$

Then (25.1) and (25.2) are satisfied identically. $\mathbf{A}$, φ are not uniquely determined by these equations since

$$\mathbf{A}' = \mathbf{A} + \nabla \Lambda$$

$$\varphi' = \varphi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}$$

where Λ is any scalar function, give the same value of **E** and **H**. We guess that **A**, φ form a 4-vector, and from the fact that

$$A_i' = A_i + \partial_i \Lambda$$

$$\varphi = \varphi - \partial_4 \Lambda$$

We see that they form a contravariant 4-vector, φ^μ. It may be noted here that when we obtain the transformation laws of the potentials, the whole problem of finding the field of a moving charge reduces to that of finding it for a static distribution, and then applying a Lorentz transformation.

Consider the equation

$$\partial_\nu \varphi_\mu - \partial_\mu \varphi_\nu = F_{\nu\mu} \tag{25.8}$$

Replacing the derivatives of the potential by the field quantities, we get

$$F_{12} = -F_{21} = H_z$$

$$F_{13} = -F_{31} = -H_y$$

$$F_{23} = -F_{32} = H_x$$

$$F_{41} = -F_{24} = -E_x$$

$$F_{42} = -F_{24} = -E_y$$

$$F_{43} = -F_{34} = -E_z$$

and writing $F_{\mu\nu}$ in matrix form we have

$$F_{\mu\nu} = \begin{array}{c|cccc} & \nu \rightarrow & & & \\ \hline \mu & 0 & H_z & -H_y & E_x \\ \downarrow & -H_z & 0 & H_x & E_y \\ & H_y & -H_x & 0 & E_z \\ & -E_x & -E_y & -E_z & 0 \end{array} \tag{25.9}$$

Raising both indices to $F^{\mu\nu}$ gives the same matrix except that the signs of E_x, E_y, E_z, are altered. If we put this value of $F^{\mu\nu}$ in (25.7) we obtain

Maxwell's equation. For example for $\mu = 1$, we have

$$\frac{\partial F^{12}}{\partial y} + \frac{\partial F^{13}}{\partial z} + \frac{1}{c}\frac{\partial F^{14}}{\partial t} = 4\pi j_x$$

which is just

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \frac{1}{c}\frac{\partial E_x}{\partial t} = 4\pi j_x$$

The other set of Maxwell's Eqs. (25.1) and (25.2) can also be written in covariant form.

$$\nabla \cdot \mathbf{H} = 0$$

written in terms of $F_{\mu\nu}$ is

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0$$

Note that the Index 4 is missing and that the indices 1, 2, 3 are cyclically permuted. Let us write an analogous equation in which the index 1 is missing.

$$\partial_2 F_{34} + \partial_3 F_{42} + \partial_4 F_{23} = 0$$

This is the same as

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{1}{c}\frac{\partial H_x}{\partial t} = 0$$

which is just

$$\left(\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E}\right)_x = 0$$

Thus (25.1) and (25.2) may be written in the form

$$\sum_{\text{cycl. perm.}} \partial_\mu F_{\lambda\nu} = 0 \tag{25.10}$$

where μ, λ, ν are all different. This is a completely antisymmetric tensor of third rank and is equivalent to a 4 vector. We can write the equation as a divergence of a tensor by introducing the dual of $F_{\mu\nu}$. The dual $G^{\alpha\beta}$ of a tensor $F_{\mu\nu}$ is defined as

$$G^{\alpha\beta} = i\varepsilon^{\alpha\beta\mu\nu} F_{\mu\nu} \tag{25.11}$$

where $\varepsilon^{\alpha\beta\mu\nu}$ is the antisymmetric unit tensor of the fourth rank. Its components vanish unless all 4 indices are different, and the nonvanishing components equal $+1$ or -1 according as $\alpha\beta\mu\nu$ form an even or an odd permutation of 1, 2, 3, 4. In terms of $G^{\alpha\beta}$ Maxwell's Eqs. (25.1) and (25.2) can be written

$$\partial_\beta G^{\alpha\beta} = 0 \tag{25.12}$$

To see this we make use of the fact that

$$\varepsilon^{\alpha\beta\mu\nu} = \varepsilon^{\alpha\mu\nu\beta} = \varepsilon^{\alpha\nu\beta\mu}$$

and hence

$$\begin{aligned}\partial_\beta G^{\alpha\beta} &= i\varepsilon^{\alpha\beta\mu\nu}\,\partial_\beta F_{\mu\nu}\\ &= \frac{i}{3}(\varepsilon^{\alpha\beta\mu\nu} + \varepsilon^{\alpha\mu\nu\beta} + \varepsilon^{\alpha\nu\beta\mu})\,\partial_\beta F_{\mu\nu}\\ &= \frac{i}{3}\varepsilon^{\alpha\beta\mu\nu}(\partial_\beta F_{\mu\nu} + \partial_\nu F_{\beta\mu} + \partial_\mu F_{\nu\beta})\end{aligned}$$

Each of the four independent values of α leads to one of the Eq. (25.10) which have already been shown to be equivalent to (25.1) and (25.2).

If we introduce the complex tensor

$$K^{\alpha\beta} = F^{\alpha\beta} + G^{\alpha\beta}$$

then the whole set of Maxwell's equations can be written

$$\partial_\alpha K^{\mu\alpha} = 4\pi j^\mu \tag{25.13}$$

The real part gives (25.3) and (25.4), and the imaginary part gives (25.1) and (25.2).

From the tensor equations we find the following transformation equations for **E** and **H**:

$$\left.\begin{aligned}\mathbf{E}'_{\|} &= \mathbf{E}_{\|}\\ \mathbf{H}'_{\|} &= \mathbf{H}_{\|}\\ \mathbf{E}'_{\perp} &= \gamma\left\{\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{H}\right\}_{\perp}\\ \mathbf{H}'_{\perp} &= \gamma\left\{\mathbf{H} - \frac{\mathbf{v}}{c}\times\mathbf{E}\right\}_{\perp}\end{aligned}\right\} \tag{25.14}$$

We have an indication here that the force on a moving charge is

$$e\left\{\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{H}\right\}$$

We shall see later that this Lorentz force can be obtained from the electrostatic force. To verify these formulae, we have

$$E'_x = F'_{14} = Q_1^\alpha Q_4^\beta F_{\alpha\beta}$$

and using the value of the matrix Q (24.1) obtained in section 24, we get

$$\begin{aligned} E'_x &= \gamma Q_4^\beta F_{1\beta} - \gamma\beta Q_4^\beta F_{4\beta} \\ &= \gamma^2 F_{14} + \gamma^2\beta^2 F_{41} \\ &= E_x \end{aligned}$$

EXERCISE 29 Check the rest of the transformation formulae for **E** and **H**.

Let us now apply the above formulae to find the field of a uniformly moving point charge, and check the results obtained in section 11. Let a charged particle be at rest in the primed system, and denote its position in this system by $\mathbf{r}'_e = (x'_e, y'_e, z'_e)$. We shall denote the point where we want the field by $\mathbf{r}'_0 = (x'_0, y'_0, z'_0)$. Then

$$\mathbf{H}' = 0$$

$$\mathbf{E}' = \frac{e(\mathbf{r}'_o - \mathbf{r}'_e)}{|\mathbf{r}_0 - \mathbf{r}_e|^3}$$

To find the field in the unprimed system, in which the particle is moving with velocity $\mathbf{v}$ along the positive x axis, we transform the two sides of the equation. For the right side,

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

For the left side

$$H_x = H'$$

$$E_x = E'_x = \frac{e(x'_0 - x'_e)}{|\mathbf{r}'_0 - \mathbf{r}'_e|^3}$$

$$\mathbf{H}_\perp = +\gamma \frac{\mathbf{v}}{c} \times \mathbf{E}'$$

$$= +\gamma e \frac{\dfrac{\mathbf{v}}{c} \times (\mathbf{r}'_0 - \mathbf{r}'_e)}{|\mathbf{r}'_0 - \mathbf{r}'_e|^3}$$

$$\mathbf{E}_\perp = \gamma E'_\perp$$

$$= \frac{e\gamma(\mathbf{r}'_0 - \mathbf{r}'_e)_\perp}{|\mathbf{r}'_0 - \mathbf{r}'_e|^3}$$

Let

$$\mathbf{R}^* = \frac{1}{\gamma}(\mathbf{r}_0' - \mathbf{r}_e')$$

$$= \left\{(x_0 - x_e + vt), \frac{1}{\gamma}(y_0 - y_e), \frac{1}{\gamma}(z_0 - z_e)\right\}$$

$$\mathbf{R} = \{(x_0 - x_e + vt), (y_0 - y_e), (z_0 - z_e)\}$$

Then

$$\mathbf{E} = \frac{e\mathbf{R}}{\gamma^2 R^{*3}} = \frac{e\mathbf{R}}{R^{*3}}(1 - \beta^2) \tag{25.15}$$

and

$$\mathbf{H} = \frac{\mathbf{v}}{c} \times \mathbf{E} \tag{25.16}$$

These equations are seen to be identical with (11.6) and (11.7). We note that $\mathbf{E}$ is perpendicular to $\mathbf{H}$, and where v is close to c we have $|\mathbf{H}| \approx |\mathbf{E}|$. Moreover, if $E_\perp$ is the magnitude of the electric field at a given distance from the particle in the particle's "equatorial plane" and $E_\parallel$ is the magnitude of the field at the same distance along the "polar axis" (i.e., the line of motion) then

$$E_\parallel = (1 - \beta^2)^{3/2} E_\perp$$

Therefore the electromagnetic field of the particle approaches more and more to that of a radiation field as v approaches c.

26 APPLICATION TO THE METHOD OF VIRTUAL QUANTA

Consider an atomic system of dimension d, bombarded by a stream of charged particles with velocity approaching that of light. When a particle passes by, the effect is much the same as if a pulse of electromagnetic radiation passed by. There is this difference, that in the case of particles there is no phase relation between the effects of different particles while there *is* a relation for the radiation field. But in such phenomena as absorption, this does not make any difference. Thus by making a Fourier analysis of the perpendicular component of the particle field, we can find what kind of electromagnetic waves it is equivalent to, and if we know what happens when the atomic system is irradiated by electromagnetic radiation of the corresponding frequencies, we can easily find what happens when the same system is bombarded by high energy charged particles.

We shall proceed to derive a formula giving the number of quanta of radiation equivalent to the passage of a single particle. From (25.15) we have

$$E_\perp = \frac{eR_\perp}{\gamma^2 R^{*3}}$$

$$= -\partial_\perp\left(\frac{e}{R^*}\right)$$

when $\partial_\perp$ stands for the derivative in the perpendicular direction. We may therefore write

$$E_\perp = \int_{-\infty}^{\infty} E_{\perp\nu} e^{2\pi i\nu t}\, d\nu$$

where

$$E_{\perp\nu} = -\partial_\perp \int_{-\infty}^{\infty} \frac{e}{R^*}\, e^{-2\pi i\nu t}\, dt$$

$$= -\partial_\perp e^{2\pi i \frac{\nu}{v}(x_0 - x_e)} \int_{-\infty}^{\infty} \frac{e^{-2\pi i\nu\left(t + \frac{x_0 - x_e}{v}\right)}\, dt}{\sqrt{v^2\left(t - \dfrac{x_0 - x_e}{v}\right)^2 + \dfrac{\varrho^2}{\gamma^2}}}$$

with $\varrho^2 = y^2 + z^2$ so that $\partial_\perp = \dfrac{\partial}{\partial\varrho}$.

Let $\xi = 2\pi\nu\left(t - \dfrac{(x_0 - x_e)}{v}\right)$. Then

$$E_{\perp\nu} = -\frac{e}{v} e^{2\pi i \frac{\nu}{v}(x_0 - x_i)} \partial_\perp \int_{-\infty}^{\infty} \frac{e^{-i\xi}\, d\xi}{\sqrt{\xi^2 + \left(\dfrac{2\pi\nu}{\gamma v}\right)^2 \varrho^2}} \tag{26.1}$$

Now

$$\int_{-\infty}^{\infty} \frac{e^{-i\xi}\, d\xi}{\sqrt{\xi^2 + \left(\dfrac{2\pi\nu}{\gamma v}\right)^2 \varrho^2}} = 2\int_{0}^{\infty} \frac{\cos\xi\, d\xi}{\sqrt{\xi^2 + \left(\dfrac{2\pi\nu}{\gamma v}\right)^2 \varrho^2}}$$

$$= 2K_0\left(\frac{2\pi\nu}{\gamma v}\varrho\right)$$

where K_0 is the Hankel function of zero order. For small values of the argument,

$$K_0(z) \approx -\ln\left(\frac{z}{2}\right)$$

and for large values of z,

$$K_0(z) \sim \sqrt{\frac{\pi}{2z}}\, e^{-z}$$

Since we are only interested in the value of this function for small value of ϱ, we shall make the approximation

$$-2K_0\left(\frac{2\pi\nu}{\gamma v}\varrho\right) \approx 2\ln\left(\frac{\pi\nu}{\gamma v}\varrho\right)$$

This will be good up to distances such that

$$\varrho \sim \frac{\gamma v}{\pi\nu}$$

and for ϱ greater than this value, we take K_0 to be zero. Thus we set

$$E_{\perp\nu} = \frac{2e}{v} e^{2\pi i\frac{\nu}{v}(x_0 - x_e)} \frac{\partial}{\partial\varrho} \ln\left(\frac{\pi\nu}{\gamma v}\varrho\right)$$

$$= \frac{2e}{v} e^{2\pi i\frac{\nu}{v}(x_0 - x_e)} \frac{1}{\varrho} \tag{26.2}$$

for ϱ up to $\approx \dfrac{\gamma v}{\pi\nu}$ and $E_{\perp\nu} = 0$ for $\varrho \geqq \dfrac{\gamma v}{\pi\nu}$.

If J_ν denotes the amount of energy per frequency range, per area, when a particle passes by at a distance ϱ, then

$$J_\nu(\varrho)\, d\nu = 2\frac{c}{4\pi} |E_{\perp\nu}|^2\, d\nu$$

The factor 2 comes from the fact that in J_ν, ν is considered to take only positive values, while in $E_{\perp\nu}$ it took both positive and negative values. Putting in the value for $E_{\perp\nu}$, we obtain

$$J_\nu(\varrho)\, d\nu = \frac{2e^2 c}{\pi v^2 \varrho^2}\, d\nu$$

Since our analysis holds only when $v \sim c$, we shall replace v by c. The total energy in frequency ν is

$$R_\nu \, d\nu = 2\pi \, d\nu \int J_\nu(\varrho) \, \varrho \, d\varrho$$

$$= 4 \frac{e^2}{c} \ln \frac{k\varrho_0}{d} \tag{26.3}$$

We have taken the limits of the integration as d and $k\varrho_0$, where $\varrho_0 = \dfrac{\gamma c}{2\pi\nu}$ d is either the size of the bombarding or the bombarded system, whichever is larger, and k is a factor of order unity. Since these values are not known exactly, they bring in an uncertainty which limits the accuracy of our calculation. However since these quantities appear as the argument of a logarithmic function, the result is rather insensitive to the error in them.

We can write

$$R_\nu \, d\nu = h\nu N_\nu \, d\nu$$

where N_ν is the number of quanta of frequency ν in the field of one particle. Thus

$$N_\nu \, d\nu = \frac{2e^2}{\pi\hbar c} \frac{1}{\nu} \ln \frac{k\gamma\lambda}{2\pi d} \, d\nu \tag{26.4}$$

since $\lambda = c/\nu$. For most problems, N_ν is of the order of a per cent. This means that about a hundred particles are required to give the same effect as a pulse of light with one quantum in each frequency range, up to a certain maximum, $\nu_{\max}$.

We will now give applications of the formula (26.4) to some specific problems:

1) *Photodisintegration of nuclei.* The cross section for disintegration of a beryllium nucleus by γ-rays is zero for γ-ray energies up to about [1.5 MeV and above this it is about 3×10^{-28} cm^2. For electrons with 2 MeV] energy we have

$$\gamma \approx 5, \quad d \approx \hbar/\text{mc} \approx 10^{-10}\,\text{cm}$$

Therefore

$$N_\nu \sim \frac{1}{100\nu}$$

$$N \sim \frac{1}{100} \int_{1{,}5\text{MeV}}^{\nu\text{MeV}} \frac{d\nu}{\nu}$$

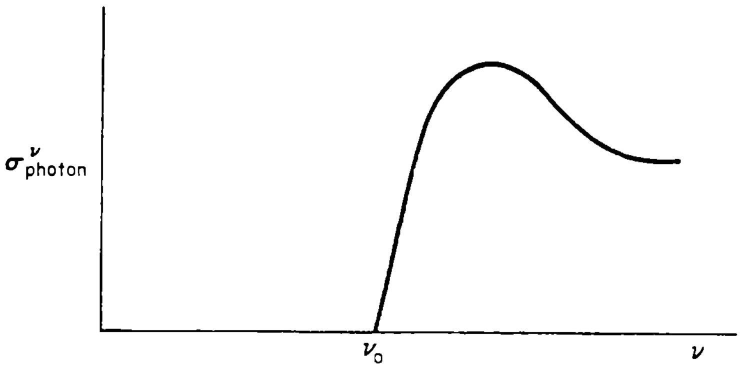
and

$$\sigma_{\text{electron}} \sim 10^{-30}\ \text{cm}^2$$

A closer estimate may be obtained if σ_{photon} is known as a function of ν. Then

$$\sigma_{\text{electron}} = \int_{\nu_0}^{E/h} \sigma^{\nu}_{\text{photon}} d\nu\, N_\nu$$

Actually, $\sigma^{\nu}_{\text{photon}}$ has the following general shape



2) *Bremsstrahlung from very high energy electrons.* This is the calculation of the cross-section for radiation by an electron accelerated in the field of a nucleus of charge Ze. The straightforward calculation is very difficult if the electron has high energy, and we resort to a trick. We make a Lorentz transformation to a system in which the electron is at rest. Then the nucleus goes by the electron at a high velocity, and to the electron, the field of the nucleus will appear as a highly contracted electromagnetic wave pulse. This wave will be scattered by the electron, and the electron will suffer a compton recoil. If we transform back to the system in which the nucleus is at rest, the recoil of the electron becomes its deflection, and the scattered pulse becomes the Bremsstrahlung.

Let ν_0 be the frequency of a virtual quantum, and ν the frequency of the corresponding scattered quantum. ν is a function of ν_0 and the angle of scattering θ. ν_r, the frequency of the radiated quantum, is the Lorentz transform of ν. If we let

$$f = \frac{h\nu_0}{mc^2} \qquad \bar{f} = \frac{h\bar{\nu}}{mc^2}$$

$$y = 1 - \cos\theta$$

$$x = \cos\theta$$

then the formula of the Compton effect is

$$\frac{1}{\bar{f}} = \frac{1}{f} + y$$

The cross section for scattering of photons by free electrons is given by the Klein–Nishina formula

$$d\sigma = \pi\, dx \frac{e^4}{m^2c^4} \frac{1}{(1+fy)^2} \left\{ 1 + x^2 + \frac{f^2y^2}{1+fy} \right\}$$

It is uncertain as to what value should be taken for d, but it turns out that the correct value is the Compton wave length $\hbar/mc$.

EXERCISE 30 Assuming $\gamma \ll 1$, calculate the cross section for Bremsstrahlung by this method, and show that it gives the Bethe–Heitler formula

$$d\sigma = \frac{4e^2}{\hbar c} \frac{Z^2e^4}{m^2c^4} \ln \frac{2\gamma(1-s)}{s} \left\{ s + \frac{4}{3} \frac{1-s}{s} \right\} ds$$

where

$$s = \frac{h\nu'_r}{\gamma mc^2}$$

3) *Pair formation by γ-rays in a nuclear field.* As a consequency of Dirac's electron theory, an electron-positron pair can be produced when two quanta of radiation collide. The cross section for pair formation when the two photons have just enough energy to create a pair is

$$\sigma = \frac{e^4}{m^2c^4}$$

Using the method of virtual quanta, we can find the cross section for pair formation when a very high energy γ-ray passes near a nucleus. We transform to a Lorentz frame in which the frequency of the γ-ray is $\nu = mc^2/h$. The nucleus is then traveling with velocity $v \sim c$, and its field will contain virtual quanta of frequency $\nu = mc^2/h$. The interaction between the γ-ray and the virtual quanta will result in pair production. It is found by this method that the probability of making pairs by a single γ-ray of energy about 10^9 e.v. in the field of a nucleus is a few per cent of that by two γ-rays of energy $\sim mc^2$.

27 APPLICATION TO THE THEORY OF THE ČERENKOV EFFECT

We saw in the last section (26.1) that the Fourier component of the transverse electric field of a charged particle moving along the x-axis has the factor

$$e^{2\pi i \frac{\nu}{v} x}$$

Thus it is equivalent to a wave motion with propagation vector

$$k_x = \frac{\nu}{v} > \frac{\nu}{c}$$

Hence

$$k^2 = k_x^2 + k_y^2 + k_z^2 > \frac{\nu^2}{c^2}$$

For an electromagnetic field in free space, $k^2 = \nu^2/c^2$, and this is the reason no matter how large the energy, a particle moving uniformly in free space cannot radiate. The field has too much momentum for its energy. However, in a dielectric the propagation vector of an electromagnetic wave satisfies the relation

$$k^2 = \frac{\nu^2 n^2}{c^2}, \quad n = \text{index of refraction}$$

and this relation can be satisfied by high energy electrons. The radiation from them has been observed by Čerenkov.

Let us consider an electron moving along the x-axis with velocity v. Then the current $\mathbf{j}$ is parallel to $\mathbf{v}$ and is given by the relation

$$j_x = \frac{ev}{c} \delta(x - vt)\, \delta(y)\, \delta(z) \tag{27.1}$$

In section 7 we derived the expression (7.11) for the total radiation emitted in frequency range $d\nu$ and in solid angle $d\Omega$. In this formula, ν is the circular frequency, and, if we write it in terms of the actual frequency, we get

$$dQ_\nu\, d\nu = d\nu\, d\Omega \frac{2\pi\nu^2}{c} \left| \iint d\mathbf{r}\, dt\, [\mathbf{j} \times \mathbf{n}]\, e^{-2\pi i \left(\nu t - \frac{\nu}{c} \mathbf{r}\cdot\mathbf{n}\right)} \right|^2 \tag{27.2}$$

The formula for the case where the system is immersed in a dielectric of refractive index n is obtained by replacing c by c/n in the two places where c occurs in the above formula. This follows from the fact that the wave equation satisfied by $\mathbf{E}$ and $\mathbf{H}$ in a medium with dielectric constant ε and permeability 1, is

$$\frac{\varepsilon}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \Delta \mathbf{E} = \frac{4\pi}{c}\frac{\partial \mathbf{j}_\perp}{\partial t}$$

so that the velocity of the propagation is $c/\sqrt{\varepsilon}$ instead of c, and from the definition of n, this equals c/n. This justifies the substitution of c/n for c in the exponential. Further, in the equation

$$\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = 0$$

the differentiation with respect to the space coordinates brings down a factor n/c instead of $1/c$ so that

$$|\mathbf{H}| = n\,|\mathbf{E}|$$

Now the Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}$$

is the same in a dielectric as in free space since the tangential components of **E** and **H** are continuous at the boundary of a dielectric, and the normal flow of energy at the boundary must be continuous. This means that n must occur as a factor in (27.2), and so we replace c by c/n there.

Let us specify the direction of the radiation by the angles φ, θ, with respect to the x-axis. Then if $\alpha = \cos\theta$, we have

$$dQ_\nu d\nu = d\nu d\varphi d\alpha \frac{2\pi\nu^2 n}{c}\left|\iint d\mathbf{r}dt[\mathbf{j} \times \mathbf{n}]\, e^{-2\pi i\left(\nu t - \frac{\nu n}{c}\mathbf{r}\cdot\mathbf{n}\right)}\right|^2$$

and using (27.1), we get

$$dQ_\nu d\nu = d\nu d\varphi d\alpha \frac{2\pi\nu^2 n}{c}\frac{e^2 v^2}{c^2}(1 - \alpha^2)\,|I|^2$$

where

$$I = \iiiint dxdydzdt\, \delta(x - vt)\,\delta(y)\,\delta(z)\, e^{-2\pi i\left(\nu t - \frac{\nu n}{c}x\alpha\right)}$$

The y and z integrations can be carried out immediately to give

$$I = \iint dxdt\delta(x - vt)\, e^{-2\pi i\left(\nu t - \frac{\nu n}{c}x\alpha\right)}$$

Then the x integration gives

$$I = \int dt e^{-2\pi i\nu t\left(1 - \frac{nv}{c}\alpha\right)}$$

If we take the integral over t from $-\infty$ to ∞, we would get

$$I = \frac{1}{\nu}\delta\left(1 - \frac{nv}{c}\alpha\right)$$

and if this value is substituted in dQ_ν and the total radiation calculated, we would get an infinite value. This is to be expected since it means that the electron radiates for an infinite time. Instead we shall take the limits of integration to be from $-T$ to $+T$ where $T \gg 1/\nu$. Then

$$I = \int_{-T}^{T} dte^{+ia\left(\alpha - \frac{c}{nv}\right)t}$$

where

$$a = 2\pi\nu\frac{nv}{c}$$

This gives

$$I = \frac{2\sin a\left(\alpha - \dfrac{c}{nv}\right)T}{a\left(\alpha - \dfrac{c}{nv}\right)}$$

and

$$dQ_\nu d\nu = d\nu d\varphi d\alpha\frac{8\pi e^2 v^2 n\nu^2}{c^3}(1 - \alpha^2)\frac{\sin^2 a\left(\alpha - \dfrac{c}{nv}\right)T}{a^2\left(\alpha - \dfrac{c}{nv}\right)^2}$$

Since $aT \gg 1$, this has a steep, narrow maximum at $\alpha = c/nv$, and is negligible everywhere else. Hence the radiation will be confined to the cone $\alpha = c/nv$.

The total radiation is given by

$$Q_\nu d\nu = d\nu\int_0^{2\pi} d\varphi\int_{-1}^{1} d\alpha\frac{8\pi e^2 v^2 n\nu^2}{c^3}(1 - \alpha^2)\frac{\sin^2 a\left(\alpha - \dfrac{c}{nv}\right)T}{a^2\left(\alpha - \dfrac{c}{nv}\right)^2}$$

$$= d\nu\frac{16\pi^2 e^2 v^2 n\nu^2}{c^3}\left(1 - \frac{c^2}{n^2 v^2}\right)\int_{-\infty}^{\infty} d\alpha\frac{\sin^2 a\left(\alpha - \dfrac{c}{nv}\right)T}{a^2\left(\alpha - \dfrac{c}{nv}\right)^2}$$

since the integrand is only appreciable at $\alpha = c/nv$. Now

$$\int_{-\infty}^{\infty} d\alpha \frac{\sin^2 a\left(\alpha - \dfrac{c}{nv}\right)T}{a^2\left(\alpha - \dfrac{c}{nv}\right)^2} = \frac{T^2}{aT}\int_{-\infty}^{\infty} dx\,\frac{\sin^2 x}{x^2}$$

$$= \frac{T}{a}\pi$$

Therefore

$$Q_\nu d\nu = d\nu\,\frac{16\pi^3 e^2 v^2 n\nu^2}{c^3}\left(1 - \frac{c^2}{n^2v^2}\right)\frac{T}{2\pi\nu\dfrac{nv}{c}}$$

$$= d\nu\,\frac{8\pi^2 e^2 v\nu}{c^2}\left(1 - \frac{c^2}{n^2v^2}\right)T$$

Thus the rate of radiation of energy of frequency ν is

$$R_\nu = \frac{4\pi^2 e^2 v\nu}{c^2}\left(1 - \frac{c^2}{n^2v^2}\right)$$

and the number of quanta of frequency ν emitted per unit path length is

$$\frac{dN_\nu}{dl} = \frac{R_\nu}{vh\nu}$$

$$= \frac{4\pi^2 e^2}{c^2 h}\left(1 - \frac{c^2}{n^2v^2}\right)$$

The number of quanta in the visible region (4×10^{14} Hz $< \nu < 10^{15}$ Hz) emitted per unit length by an electron with $v/c = 0.95$ passing through water ($n = 1.3$) is

$$\frac{dN}{dl} = \frac{2\pi}{137}\,\frac{6 \cdot 10^{14}}{3 \cdot 10^{10}}\left(1 - \frac{1}{(1.3)^2 \cdot (0.95)^2}\right)$$

$$\sim 400 \text{ quanta per cm}$$

28 TRANSFORMATION OF ENERGY AND MOMENTUM

We shall prove that if **p** is the momentum and E the energy of a charged particle, then $(\mathbf{p}, E/c)$ form a 4 vector, and it will follow from this that

$$E^2 - c^2p^2 = \text{invariant}$$

We shall also show that

$$E = \frac{m_0c^2}{\sqrt{1-\beta^2}}$$

$$\mathbf{p} = \frac{m_0\mathbf{v}}{\sqrt{1-\beta^2}}$$

where m_0 is the rest mass of the particle.

We have seen in section 5 that for an electromagnetic field, we have the conservation laws

$$\frac{dW}{dt} + \operatorname{div}\mathbf{S} = -c(\mathbf{E}\cdot\mathbf{j}) \tag{28.1}$$

$$\frac{d\mathbf{G}}{dt} + \operatorname{div}\mathbf{T} = -\{\varrho\mathbf{E} + \mathbf{j}\times\mathbf{H}\} \tag{28.2}$$

where

$$W = \frac{E^2 + H^2}{8\pi}$$

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}$$

$$\mathbf{G} = \frac{1}{4\pi c}\mathbf{E}\times\mathbf{H}$$

$$T_{ij} = -\frac{1}{4\pi}\{E_iE_j + H_iH_j - \frac{1}{2}\delta_{ij}(E^2 + H^2)$$

It seems that (28.1) and (28.2) should be expressible in covariant form, and we shall try to do this. We want to write the equations in terms of the quantities

$$j^\mu = (\mathbf{j}, \varrho)$$

$$F_{\mu\nu} = (\mathbf{E}, \mathbf{H})$$

Let

$$f_i = \varrho E_i + [\mathbf{j}\times\mathbf{H}]_i$$

and consider

$$f_\mu = F_{\mu\nu} j^\nu \tag{28.3}$$

We have

$$f_1 = H_z j_y - H_y j_z + E_x \varrho = f_x$$

$$f_2 = -H_z j_x + H_x j_z + E_y \varrho = f_y$$

$$f_3 = H_y j_x - H_x j_y + E_z \varrho = f_z$$

$$f_4 = -E_x j_x - E_y j_y - E_z j_z = -\mathbf{E} \cdot \mathbf{j}$$

Next consider the equation

$$4\pi T^\mu_\nu = F^{\mu\alpha} F_{\nu\alpha} - \frac{1}{4} \delta^\mu_\nu F^{\alpha\beta} F_{\alpha\beta} \tag{28.4}$$

We have

$$4\pi T^4_4 = F^{41} F_{41} + F^{42} F_{42} + F^{43} F_{43} - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta}$$

$$= -E_x^2 - E_y^2 - E_z^2 - \frac{1}{2}(H^2 - E^2)$$

$$= -\frac{E^2 + H^2}{2}$$

and hence

$$T^4_4 = -W$$

Similarly

$$4\pi T^4_1 = F^{42} F_{12} + F^{43} F_{13}$$

$$= E_y H_z - E_z H_y$$

$$= [\mathbf{E} \times \mathbf{H}]_x$$

whence

$$T^4_1 = \frac{1}{c} S_x = c G_x$$

Finally,

$$4\pi T^1_1 = F^{12} F_{12} + F^{13} F_{13} + F^{14} F_{14} - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta}$$

$$= H_z^2 + H_y^2 - E_x^2 - \frac{1}{2}(H^2 - E^2)$$

$$= \frac{1}{2}\{-E_x^2 + E_y^2 + E_z^2 - H_x^2 + H_y^2 + H_z^2\}$$

which gives

$$T_1^1 = T_{xx}$$

and

$$4\pi T_1^2 = F^{23}F_{13} + F^{24}F_{14}$$

$$= -H_x H_y - E_y E_x$$

which gives

$$T_1^2 = T_{yx}$$

etc. If the other components are calculated in a similar manner, we find

$$4\pi T^\mu_\nu = \begin{array}{c|cccc} \mu\downarrow\ \nu\rightarrow & & & & \\ \hline & \frac{1}{2}(E^2+H^2) - E_x^2 - H_x^2 & -E_xE_y - H_xH_y & -E_zE_x - H_zH_x & -E_yH_z + E_zH_y \\ & -E_xE_y - H_xH_y & \frac{1}{2}(E^2+H^2) - E_y^2 - H_y^2 & -E_yE_z - H_yH_z & -E_zH_x + E_xH_z \\ & -E_zE_x - H_zH_x & -E_yE_z - H_yH_z & \frac{1}{2}(E^2+H^2) - E_z^2 - H_z^2 & -E_xH_y + E_yH_x \\ & E_yH_z - E_zH_y & E_zH_x - E_xH_z & E_xH_y - E_yH_x & -\frac{1}{2}(E^2+H^2) \end{array} \quad (28.5)$$

It can also be written in the form

$$T^\mu_\nu = \begin{array}{c|cccc} \mu\downarrow\ \nu\rightarrow & & & & \\ \hline & T_{xx} & T_{xy} & T_{zx} & -cG_x \\ & T_{xy} & T_{yy} & T_{yz} & -cG_y \\ & T_{zx} & T_{yz} & T_{zz} & -cG_z \\ & cG_x & cG_y & cG_z & -W \end{array} \quad (28.6)$$

T^μ_ν is traceless since

$$4\pi T^\mu_\mu = F^{\mu\alpha}F_{\mu\alpha} - \frac{1}{4}\cdot 4F^{\alpha\beta}F_{\alpha\beta} = 0$$

Hence for an isotropic distribution of radiation,

$$T_1^1 = T_2^2 = T_3^3 = -\frac{1}{3}T_4^4$$

It is now easy to see that the equation

$$\partial_\mu T^\mu_\nu = -f_\nu \tag{28.7}$$

is the covariant expression of (28.1) and (28.2). For $\nu = 1$ we have

$$\sum^3 \frac{\partial}{\partial x_j} T_{ij} + \frac{\partial G_1}{\partial t} = -f_1$$

and for $\nu = 4$,

$$-\frac{1}{c}\nabla \cdot \mathbf{S} - \frac{\partial W}{c\,\partial t} = \mathbf{E} \cdot \mathbf{j}$$

Thus (28.7) is the covariant form of the conservation equations.

Let us suppose that we have some electromagnetic radiation in a finite region of space so that $\mathbf{E}$ and $\mathbf{H}$ exist in this region but are zero everywhere else. Then the integrals

$$E = \int T^4_4 \, d\mathbf{r}$$

$$cp_i = \int T^4_i \, d\mathbf{r}$$

exist. We shall show that $(\mathbf{p}, E/c) \equiv p_\mu$ is a covariant 4-vector if there are no charges in the region where the fields are not zero.

The proof is as follows: We have

$$\partial_\mu T^\mu_\nu = 0$$

everywhere, and

$$T^\mu_\nu = 0$$

on the boundary of the region under consideration. Let A^μ be a vector such that

$$\partial_\mu A^\mu = 0$$

everywhere and

$$A^\mu = 0$$

on the boundary. Applying Gauss's theorem to the 4-dimensional divergence of a 4-vector, we obtain from

$$\iiiint \partial_\mu A^\mu \, dx' \, dx^2 \, dx^3 \, dx^4 = 0$$

that

$$\iiint A_{\text{normal}} \, dS = 0$$

where S is the 3-dimensional surface in the 4-dimensional volume. Let us choose this volume to be a cylinder parallel to the x^4 axis. Then the base is the 3-dimensional volume in ordinary space. The cylinder is to be bounded by a section x^4 = constant and by another section x^4 = constant, where $x^{4\prime}$ denotes the time measured by an observer on a coordinate system moving with respect to the original system. The normal components are $-A^4$ and $(A^4)'$ on these surfaces, and zero on the walls of the cylinder. Hence we have

$$-\iiint A^4\, dx^1\, dx^2\, dx^3 + \iiint (A^4)'\,(dx^1)'\,(dx^2)'\,(dx^3)' = 0$$

That is

$$\iiint A^4\, dx^1\, dx^2\, dx^3 = \text{invariant}$$

Now if we let

$$b^\nu T^\mu_\nu = A^\mu$$

where b^ν is an arbitrary, constant vector, we see that the conditions on A^μ are satisfied, and hence

$$b^\nu \iiint T^4_\nu\, d\mathbf{r} = \text{invariant}$$

and since b^ν is arbitrary, we may conclude that

$$\iiint T^4_\nu\, d\mathbf{r}$$

form a 4-vector.

Let w and Э denote the momentum and energy densities of a charged body. We know that at low velocities, they satisfy the following equations

$$\frac{dw}{dt} = \mathbf{f}$$

$$\frac{d\text{Э}}{dt} = \mathbf{f}\cdot\mathbf{v}$$

We shall generalize the meanings of w and Э so that these equations hold for all velocities. Let us integrate these equations over the whole body.

Then

$$\frac{d\mathbf{p}}{dt} = \int \mathbf{f}\, d\mathbf{r} = \mathscr{F}$$

$$\frac{dE}{dt} = \int (\mathbf{f} \cdot \mathbf{v})\, d\mathbf{r}$$

where $\mathbf{p}$ and E are the total momentum and energy of the body. Now

$$d\mathbf{r} = d\mathbf{r}_0\sqrt{1 - \beta^2} = \frac{1}{\gamma}\, d\mathbf{r}_0$$

$$\gamma \frac{d}{dt} = \frac{d}{ds}$$

where $d\mathbf{r}_0$ is the proper volume element and s is the proper time. Proper quantities are quantities measured in the co-moving system, and hence are invariants. Thus

$$\frac{d\mathbf{p}}{ds} = \int \mathbf{f}\, d\mathbf{r}_0 = \mathbf{F}$$

$$\frac{dE}{ds} = \int (\mathbf{f} \cdot \mathbf{v})\, d\mathbf{r}_0$$

We have seen that $\mathbf{f}$ and $-(\mathbf{E} \cdot \mathbf{j})$ together form a covariant 4 vector, and since

$$(\mathbf{E} \cdot \mathbf{j}) = \frac{1}{c}(\mathbf{f} \cdot \mathbf{v})$$

we see that

$$\frac{d\mathbf{p}}{ds} \text{ and } \frac{1}{c}\,\frac{dE}{ds}$$

form a contravariant 4-vector F^μ. This means that, with a proper choice of energy zero point, $\mathbf{p}$ and E/c vary contravariantly, and we can write

$$p^\mu = \left(\mathbf{p}, \frac{E}{c}\right)$$

as a contravariant 4-vector.

We have introduced 3 different forces and we shall summarize their properties:

1) $f^\mu = (\mathbf{f}, \mathbf{E} \cdot \mathbf{j})$, where $\mathbf{f}$ is the force density, is a contravariant 4-vector and equals the time rate of change of momentum density for $\mu = 1, 2, 3$ and $1/c$ times the time rate of change of the energy density for $\mu = 4$.

2) $F^\mu = \int f^\mu \, d\mathbf{r}_0$, where $d\mathbf{r}_0$ is an element of proper volume, is a contravariant 4-vector which equals the proper time rate of change of momentum for $\mu = 1, 2, 3$ and $1/c$ times the proper time rate of change of energy for $\mu = 4$.

3) $\mathscr{F} = \sqrt{1 - \beta^2}\mathbf{F}$ is the actual force and equals the time rate of change of momentum. It is not a part of a 4-vector.

From the fact that $\mathbf{p}$ and E/c form a contravariant 4-vector it follows that

$$p_\mu p^\mu = \text{invariant}$$

Now suppose we transform to a system such that $\mathbf{p}$ changes by $\Delta\mathbf{p}$. Then

$$\sum \left\{ p_i \, \Delta p_i - \frac{E}{c^2} \frac{\partial E}{\partial p_i} \Delta p_i \right\} = 0$$

It follows from the Hamiltonian theory of dynamics that

$$\frac{\partial E}{\partial p_i} = v_i$$

a relation which is also necessary for the wave-mechanical interpretation of matter. Therefore

$$p_i = \frac{E}{c^2} v_i$$

Now we know that for v/c small

$$p_i \approx m_0 v_i$$

Hence

$$\frac{E}{c^2} \approx m_0$$

and

$$\frac{E^2}{c^2} - p^2 = m_0 c^2 \tag{28.8}$$

Since the quantity in the final equation is an invariant we may now generalize to arbitrary velocities:

$$\frac{E^2}{c^2} - \frac{E^2}{c^4} v^2 = m_0^2 c^2$$

$$E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} = \gamma m_0 c^2 \tag{28.9}$$

$$\mathbf{p} = \frac{m_0 \mathbf{v}}{\sqrt{1 - v^2/c^2}} = \gamma m_0 \mathbf{v} \tag{28.10}$$

Thus if e is the total charge of the particle, the equations of motion are

$$\frac{d}{dt}(m_0 \gamma \mathbf{v}) = e\left\{\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{H}\right\} \tag{28.11}$$

$$\frac{d}{dt}(m_0 \gamma c^2) = e\,\mathbf{E} \cdot \mathbf{v} \tag{28.12}$$

EXERCISE 31 In section 6 we found the rate of radiation from a charge in the proper reference frame. That is

$$\frac{dE_0}{dt_0} = \frac{2}{3}\frac{e^2}{c^3}\,\dot{\mathbf{v}}_0^2$$

From the transformation properties of E and t, we see that dE/dt is an invariant. Hence

$$\frac{dE}{dt} = \frac{2}{3}\frac{e^2}{c^3}\,\dot{\mathbf{v}}_0^2$$

Calculate $\dot{\mathbf{v}}_0^2$ in terms of the variables of the moving system, and show the rate of radiation from a moving charge calculated in this way agrees with (13.1).

For a particle in a uniform magnetic field, γ is constant and

$$\dot{\mathbf{v}} = \frac{e}{m_0 \gamma c}\mathbf{v} \times \mathbf{H}$$

Thus for a plane motion, the trajectory is a circle of radius ϱ where

$$\frac{1}{\varrho} = \frac{eH}{m_0 \gamma c v}$$

This relation may also be expressed in the useful form

$$H\varrho = \frac{cp}{e} \tag{28.13}$$

The transformation formulae for the mass and momentum of a particle can also be obtained from purely kinematical reasoning. We consider the collision of two particles, and demand that in the collision, both energy and momentum be conserved for an observer in any system. The required result can be obtained by considering the simple case of the head-on collision of two perfectly elastic particles with equal rest mass. Then there is one reference system in which the particles have equal and opposite velocities, say u and $-u$ before the collision, and after collision separate with velocities $-u$ and u. For an observer moving with velocity $-V$ with respect to this system and parallel to the direction of motions of the particles, the initial velocities v_1 and v_2 are given by

$$v_1 = \frac{u + V}{1 + \dfrac{uV}{c^2}}$$

$$v_2 = \frac{-u + V}{1 - \dfrac{uV}{c^2}}$$

In the first reference frame, there is some instant during the collision when the two particles are in contact and are at rest. Hence since the effective (or inertial) mass m can depend only on the absolute value of the velocity, we have

$$2m(u) = M_0$$

$$m(u)\, u + m(u)\,(-u) = 0$$

In the second reference frame, at some instant the two bodies are together and have a common velocity V; hence

$$m(v_1) + m(v_2) = M$$

$$m(v_1)\, v_1 + m(v_2)\, v_2 = MV$$

In a sense, we are defining mass by these equations in such a way that it is conserved. We are considering mass instead of energy as it is more convenient to do so, and we shall show later the connection between mass defined in this way, and energy. We now have

$$m(v_1)\,v_1 + m(v_2)\,v_2 = \{m(v_1) + m(v_2)\}\,V$$

$$u\left\{\frac{m(v_1)}{1+\dfrac{uV}{c^2}} - \frac{m(v_2)}{1-\dfrac{uV}{c^2}}\right\} + V\left\{\frac{m(v_1)}{1+\dfrac{uV}{c^2}} + \frac{m(v_2)}{1-\dfrac{uV}{c^2}}\right\}$$

$$= V\{m(v_1) + m(v_2)\}$$

This equation is satisfied by

$$\frac{m(v_1)}{m(v_2)} = \frac{1+\dfrac{uV}{c^2}}{1-\dfrac{uV}{c^2}}$$

since if we write

$$\frac{m(v_1)}{1+\dfrac{uV}{c^2}} = \frac{m(v_2)}{1-\dfrac{uV}{c^2}} = k$$

and substitute in the equation, we get

$$u(k-k) + V(k+k) = V(k)\left(1 + \frac{uV}{c^2} + 1 - \frac{uV}{c^2}\right)$$

or

$$2Vk = 2Vk$$

Now

$$1 - \frac{v_1^2}{c^2} = 1 - \frac{(V+u)^2}{c^2\left(1+\dfrac{Vu}{c^2}\right)^2}$$

$$= \frac{c^2 + 2Vu + \dfrac{V^2u^2}{c^2} - V^2 - 2Vu - u^2}{c^2\left(1+\dfrac{Vu}{c^2}\right)^2}$$

$$= \left(1 - \frac{u^2}{c^2}\right)\left(1 - \frac{V^2}{c^2}\right)\Big/\left(1 + \frac{Vu}{c^2}\right)^2$$

Therefore

$$1 + \frac{Vu}{c^2} = \sqrt{\frac{\left(1 - \frac{u^2}{c^2}\right)\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \frac{v_1^2}{c^2}\right)}}$$

By changing the sign of V, we interchange v_1 and v_2, and hence

$$1 - \frac{Vu}{c^2} = \sqrt{\frac{\left(1 - \frac{u^2}{c^2}\right)\left(1 - \frac{V_2^2}{c^2}\right)}{\left(1 - \frac{v_2^2}{c^2}\right)}}$$

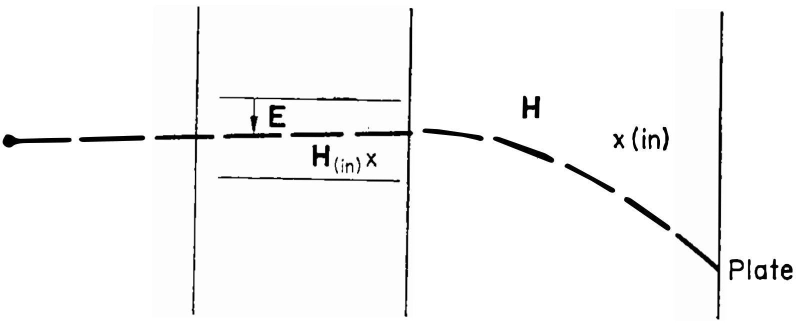
Therefore

$$\frac{m(v_1)}{m(v_2)} = \frac{\sqrt{1 - \frac{v_2^2}{c^2}}}{\sqrt{1 - \frac{v_2^2}{c^2}}}$$

Since this must hold for all u and V, we have

$$m(v) = \frac{m(0)}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m(0)$$

This functional dependence of mass on velocity was first verified ex-



perimentally by Bucherer. A combination of a uniform electric field **E** and magnetic field **H**, which are perpendicular to each other and to the direction of motion of the electrons, selects electrons with a definite velocity v given by

$$\frac{Hv}{c} = E$$

The momentum of these particles is then found by measuring the deflections in the same magnetic field. We have from (28.13)

$$eH\varrho = mvc$$

and therefore

$$m = \frac{e\varrho H^2}{c^2 E}$$

The difficulty in the experiment is to prevent electrons which are scattered at the slits and in the condenser from coming to the plate.

29 INERTIA AND ENERGY

Let us now consider the relation between mass and energy. By mass here we mean inertial mass. We shall show that for a consistent description, by different observers, of an emission process, we must have

$$\Delta m = \frac{\Delta E}{c^2} \tag{29.1}$$

The question whether we can write

$$m = \frac{E}{c^2}$$

depends on whether we can convert the whole mass of a system into some form of energy.

Consider some emission process, say the radiation from a hot body, or the β emission from a radioactive nucleus. We shall assume isotropic radiation in the rest system. Then

$$\delta\mathbf{p} = 0$$

If the process is observed from a moving system, we have

$$\delta E' = \gamma(\delta E + v\,\delta p) = \gamma\,\delta E$$

$$\delta p' = \gamma\,\delta p + \frac{v}{c^2}\gamma\,\delta E = +\frac{v}{c^2}\,\delta E'$$

The change in p' means a change either in m or v, but since v is constant, m must change.

$$\delta p' = \delta(m'v) = v\,\delta m' = v\frac{\delta E'}{c^2}$$

and hence

$$\delta m' = \frac{\delta E'}{c^2} \tag{29.1}$$

It must be pointed out here that the mass considered above is inertial mass, and the mass usually measured is gravitational mass. However, the Eötvös experiment has shown that these two masses are equivalent to a very high degree of precision.

The relation (29.1) between energy and mass has been checked experimentally in several cases. We know that positrons have the same rest mass as electrons and at low velocities the former combine with electrons to form γ-rays. The energy of the γ-rays have been measured and shown to agree with the value predicted by the theory

$$\nu = \frac{m_0 c^2}{h}$$

to about one per cent. There is no factor of 2 since two particles of mass m_0 give rise to two quanta of this energy. This is necessary in order that momentum be conserved. We note that in this process the whole mass is converted to radiation energy. The inverse process does not give a good check since the curve giving the relation between the γ-ray energy and the number of particles produced does not have a sharp threshold at the minimum energy required for pair production.

The reaction

$$\mathrm{Li}^7 + H^1 \rightarrow \begin{cases} 2\ \mathrm{He}\ (8\ \mathrm{MeV}) \\ \gamma\ (17.6\ \mathrm{MeV}) + 2\ \mathrm{He}\ (100\ \mathrm{KeV}) \end{cases}$$

has given quite an accurate check of (29.1).

30 CONSIDERATIONS IMPORTANT FOR THE QUANTUM THEORY

In this section we shall give a few considerations to show how the classical electromagnetic theory developed in this course has to be modified when quantum mechanical effects are considered.

I We have seen in section 18 that if $H(p, x)$ is the Hamiltonian function for a particle with charge e when there is no external field acting on it, then its expression when there is a field is given by

$$H' = e\varphi + H\left(p - \frac{e}{c}A, x\right)$$

so that

$$\mathbf{p} = m\dot{\mathbf{x}} + \frac{e}{c}\mathbf{A}$$

and we have also

$$\frac{d}{dt}(m\dot{\mathbf{x}}) = \mathscr{F}$$

where $\mathscr{F}$ is the Lorentz force. From relativistic considerations, we have from (28.8)

$$E = H = c\sqrt{m_0^2c^2 - p^2}$$

$$H' = e\varphi + c\sqrt{m_0^2c^2 + \left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2}$$

$$\dot{x}_i = \frac{\partial H'}{\partial p_i} = \frac{c\left(p_i - \frac{e}{c}A_i\right)}{\sqrt{m_0^2c^2 + \left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2}}$$

$$v^2 = \frac{c^2\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2}{m_0^2c^2 + \left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2}$$

$$\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 = \frac{m_0^2v^2}{1 - \frac{v^2}{c^2}}$$

$$\dot{\mathbf{x}} = \frac{c\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)}{\sqrt{m_0^2c^2 + \frac{m_0^2v^2}{1 - \frac{v^2}{c^2}}}}$$

$$\mathbf{p} = m_0\gamma\dot{\mathbf{x}} + \frac{e}{c}\mathbf{A}$$

Also

$$\frac{d}{dt}(m_0\gamma\dot{\mathbf{x}}) = \mathscr{F}$$

Now since $\mathbf{p}$, E/c and $\mathbf{A}$, φ both form contravariant 4-vectors, any linear combination of them also forms one, and hence

$$c^2\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 - (E - e\varphi)^2 = \text{invariant}$$

We find the value of this invariant by setting $E = H'$ and using the expression for H' above. This gives

$$c^2\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 - (E - e\varphi)^2 = -m_0^2c^4$$

Now this equation has not only the solution

$$E = H' = e\varphi + c\sqrt{m_0^2c^2 + \left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2}$$

but also the solution

$$E = H'' = e\varphi - c\sqrt{m_0^2c^2 + \left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2}$$

and this Hamiltonian gives the same equation of motion as H' if the signs of e and p are changed. Thus H'' corresponds to the Hamiltonian of a particle with the opposite charge and negative mass. In classical physics we can say that this does not correspond to any reality and throw this solution away; in quantum mechanics, however, it is connected with the theory of the positron. These anomalous solutions (H'') are the classical origin of the phenomenon of pair production.

II The oscillations of a plane wave are given by the factor

$$e^{2\pi i(\mathbf{k}\cdot\mathbf{r} - \nu t)}$$

Since the phase must be independent of the motion of the observer, we have

$$\mathbf{k}\cdot\mathbf{r} - \nu t = \text{invariant}$$

and since $\mathbf{r}$, ct form a contravariant 4-vector, $\mathbf{k}$, ν/c must also form a contravariant 4-vector. Thus

$$\frac{E}{c} = \text{const}\, x\frac{\nu}{c}$$

$$\mathbf{p} = \text{const}\, xk$$

are covariant relations. DeBroglie wrote

$$E = h\nu, \qquad \mathbf{p} = h\mathbf{k}$$

According to the quantum theory any system with well-defined E and $\mathbf{p}$ has connected with it well-defined values of ν and $\mathbf{k}$. This means that in making a Lorentz transformation, we get the same result whether we use the Doppler effect for ν or the transformation formula for E.

III We know from section 3 that any radiation field may be expanded in terms of plane waves. Planck showed that plane waves can only be excited with integral multiples of a discrete energy. Thus

$$E_{\mathbf{k},\,\nu} = Nh\nu$$

where N is an integral number. Moreover Compton showed that plane waves have momentum

$$\mathbf{p}_{\mathbf{k},\,\nu} = Nh\mathbf{k}$$

The energy E_k in the k^{th} Fourier component of a field oscillator is given by

$$E_k = \frac{1}{8\pi}\,(E_k^2 + H_k^2)$$

and if A_k is this component of the vector potential, then since

$$\mathbf{H} = \nabla \times \mathbf{A}$$

we have

$$E_k = \frac{1}{8\pi}\,(E_k^2 + 4\pi^2k^2A_k^2)$$

Let us compare this expression with the corresponding expression for a mechanical oscillator of unit mass

$$E = \frac{1}{2}\,(p^2 + \omega^2q^2)$$

We can make the following correspondence

$$E_k \leftrightarrow p$$

$$A_k \leftrightarrow q$$

$$2\pi k \leftrightarrow \omega$$

For the mechanical oscillator, no quantum mechanical effects are observed when it is highly excited. Thus in any phenomenon where a large number of quanta of radiation are present, we expect classical theory to hold. Roughly, this means that classical electromagnetic theory holds for low frequencies.

Let us see how the classical theory breaks down. In the quantum theory, we have the complementary relation between position and momentum of a particle

$$\Delta p\, \Delta q \geqq \hbar$$

For a well defined **p**, **q** is spread out, and for well defined **q**, **p** is spread out. Making the correspondences, we have the complementarity

$$\Delta E_k \, \Delta A_k \geqq \hbar c$$

where the factor c comes in from the consideration of units. Since A_k gives rise to a magnetic field $\mathbf{H}_k \perp$ to $\mathbf{E}_k$, this expresses a complementarity between **E** and the component of **H** perpendicular to it. Also, if the energy E of a harmonic oscillator is fixed, p and q fluctuate. Similarly, if $E_{h\nu}$ is fixed, **E** and **H** fluctuate.

In classical theory, it is assumed that **E** and **H** can be measured with arbitrary precision. This is not possible in quantum theory. In the actual measurement of **E**, we have to take a certain volume and take a certain interval of time T and take the average. Thus

$$\bar{\mathbf{E}} = \frac{1}{VT} \int_V \int_T d\mathbf{r} \, dt \, \mathbf{E}(\mathbf{r}, t)$$

One method is to take a charged body and measure its momentum at the beginning and at the end of the time interval T. Then

$$\mathbf{p}(T) - \mathbf{p}(0) = \varrho V \bar{\mathbf{E}} T$$

In order that the body stay near one position, we must take the mass large, and in order that its own field not modify the external field which we want to measure, we must take ϱ very small. In classical theory these conditions can be fulfilled, and we can make the test bodies so feeble that we can measure **E** and **H** at neighboring points without disturbing each other. In quantum theory, however, this is not possible since each momentum measurement brings in an uncertainty in the position such that

$$\delta p \geqq \hbar/\delta x$$

$$\varrho V \, \delta E T \geqq \hbar/\delta x$$

Thus

$$VT \delta E \, \delta x \geqq \hbar/\varrho$$

and for $V, T, \delta E, \delta x$ to be small, ϱ must be large. In fact, if the measurement is to have any meaning, we must have

$$\delta x \ll V^{1/3}$$

If ϱ is made large, then the self-field of the test body alters the external field. However, we can compensate this effect by having a similar body charged with equal and opposite charge, and fixed in space at the place where the field is being measured. Since there is an uncertainty δx in the position of the test body, its field will not be entirely cancelled out, but there will be a dipole moment proportional to δx. Since δx is unknown the force on the test body which is proportional to δx is also unknown. However, the constant of proportionality can be calculated, and if the test body is attached to the compensating body by a spring with an elastic constant equal to this value, then the uncertainty in position will give no net force on the test body, and the field can be measured to arbitrary precision by using a high enough charge density.

The dipole moment disturbs the value of **H** nearby, and so gives complementarity between **E** and **H**. Now a current in the x direction only gives a magnetic field in the yz plane; thus the complementarity is between the perpendicular components of **E** and **H**. The parallel components are not complementary. It can also be shown that two measurements of the field do not disturb each other if the one lies in the "absolute elsewhere" (Minkowski diagram) of the other. This is to be expected since then no electromagnetic disturbance which originates at the position and time of the measurement can reach the place of the other measurement at the time when this measurement is being made. Another result which can be proved is that if there are a large number of quanta such that

$$k \sim V^{-\frac{1}{3}}; \quad \nu \sim T^{-1}$$

then the fields due to these quanta of radiation can be measured to a small fraction of their values.

As an illustration of the complementarity between **E** and energy, let us consider the photoelectric effect. It is found that no matter how low the intensiv of the radiation, the electrons are always ejected with energy

$$E_e = h\nu - I$$

where I is the binding force of the electron to the atom. This ejection is due to the force

$$e\mathbf{E}$$

and we would expect classically that since **E** decreases with decreasing intensity, we will get a smaller value of E_e. However, due to complementarity, if the energy of the radiation is well defined, the field fluctuates violently, and though it takes on large values less frequently for lower intensities, the energy with which the electrons come out is the same, though their number is proportional to the intensity.

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